MIDTERM 2

Solutions

MTH 164 Summer Session B

Monday July 27, 2020

Instructions

- **ABSOLUTELY NO COLLABORATION IS ALLOWED ON THIS EXAM**. You are not to communicate in any way with your fellow students during this exam.
- You must justify all your work completely. No credit will be given to answers without justification.
- You may look at your textbook and notes during the exam. You may use theorems from the textbook without proof but you must prove results stated in exercises or examples if you use them.
- You will write your solutions to the below problems on paper, or type them using Latex. If you are writing your solutions on paper, please scan them using a phone app or scanner. **Write one problem per page** and clearly label which problem goes with what work.
- You have **1 hour and 15 minutes after the time you recieve the exam pdf document** to complete this exam. You will have 15 minutes to then send me your solutions in an email. The total time from when you receive the exam to when I receive your completed work is one hour and thirty minutes.
- Please send your solutions in pdf format. Pictures will be accepted in an emergency if your scanning methods fail you. **Put your name in the filename**. Send your solutions in the order they appear in this exam.

Problem 1

a) Find the limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^3y - xy^3}{x^2+y^2}$$

b) Show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

a) $\lim_{(x,y)\to(0,0)} \frac{x^3y - xy^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2}$

Note $\frac{x^2-y^2}{x^2+y^2} \le 1$ since $x^2-y^2 \le x^2 \le x^2+y^2$

$\Rightarrow \left| \lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} \right| \le \left| \lim_{(x,y)\to(0,0)} xy \right| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3y - xy^3}{x^2+y^2} = 0$

by the squeeze theorem

b) Let $y = x^3$. We have $\lim_{(x,x3)\to(0,0)} \frac{x^3 \cdot x^3}{x^6 + x^6} = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$

But if $y = x$, we have $\lim_{(x,x)\to(0,0)} \frac{x^4}{x^6+x^2} = \lim_{x\to 0} \frac{x^2}{x^4+1} = 0$

and $0 \neq 1/2$ so the limit does not exist.

Problem 2

a) The helix $\vec{r}(t) = \langle \cos(\pi t/2), \sin(\pi t/2), t \rangle$ intersects the sphere $x^2 + y^2 + z^2 = 2$ at two points. Find those two points.

b) What is the equation for the tangent plane of the sphere at the above points of intersection?

a) Plugging in $x = \cos(\pi t/2)$, $y = \sin(\pi t/2)$ & $z = t$ to the sphere equation:

$$\cos^2(\pi t/2) + \sin^2(\pi t/2) + t^2 = 1 + t^2 = 2 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

$\Rightarrow$ the two points are $P_1 = (0, 1, 1)$ and $P_2 = (0, -1, -1)$

b) Let $F = x^2 + y^2 + z^2 - 2$. Then $\nabla F = \langle 2x, 2y, 2z \rangle$.

$\nabla F|_{P_1} = \langle 0, 2, 2 \rangle$ & $\nabla F|_{P_2} = \langle 0, -2, -2 \rangle$ so the plane equation is

$$\pm 2(y \mp 1) \pm 2(z \mp 1) = 0$$

Problem 3

Find the critical points of the function $f(x, y) = x^2 - x + \cos(xy)$. You do not have to classify them.

$f_x = 2x - 1 - y\sin(xy) = 0$

$f_y = -x\sin(xy) = 0 \Rightarrow x = 0$ <u>or</u> $\sin(xy) = 0$.

If $x = 0$, $f_x = 1 - y\sin(0) = 0 \Rightarrow 1 = 0 \Rightarrow$ contradiction. $\Rightarrow x \neq 0$

$\Rightarrow \sin(xy) = 0 \Rightarrow f_x = 2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = 1/2$

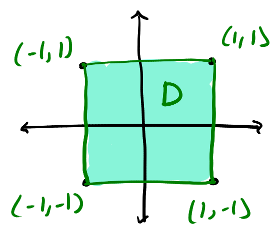
$\sin(\frac{y}{2}) = 0 \Rightarrow y/2 = \pi n$ for all $n \in \mathbb{Z} \Rightarrow y = 2\pi n$, $n \in \mathbb{Z}$

the critical points are at $(\frac{1}{2}, 2\pi n)$

Problem 4

Show that the point $(0, 0)$ is a critical point of the function $f(x, y) = x^4 + 6y^2 - 4xy^3 - 1$ and prove that it is a local minimum.

$$\left.\begin{aligned} f_x &= 4x^3 - 4y^3 = 0 \\ f_y &= 12y - 12xy^2 = 0 \end{aligned}\right\}$$ Clearly these equations are satisfied by the point $(0,0)$, so $(0,0)$ is a critical point.



Now, $f(0,0) = -1$

Claim: $f(x,y) \geq -1 \quad \forall \quad (x,y) \in D = \{(x,y) : |x| \leq 1, |y| \leq 1\}$

if & only if.

$$x^4 + 6y^2 - 4xy^3 - 1 \geq -1$$

$$\Leftrightarrow \quad x^4 + 6y^2 - 4xy^3 \geq 0$$

But $x^4 + 6y^2 - 4xy^3 \geq 6y^2 - 4xy^3$ since $x^4 \geq 0$

$\Rightarrow$ It suffices to show that $6y^2 - 4xy^3 \geq 0 \quad \forall \quad (x,y) \in D$.

We have: $$6y^2 - 4xy^3 \geq 0$$

$$\Leftrightarrow \quad 2y^2(3 - 2xy) \geq 0$$

$$\Leftrightarrow \quad (3 - 2xy) \geq 0 \quad \Leftrightarrow \quad \tfrac{3}{2} \geq xy$$

But if $|x| \leq 1$ & $|y| \leq 1$, $|xy| \leq 1 \leq \frac{3}{2}$ $\square$

Problem 5

Find the extreme values of $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 = z^2$, $x + y + z = 24$

$\nabla f = \langle 0,0,1 \rangle$ $\quad g = x^2+y^2-z^2 \quad h = x+y+z$

$\nabla g = \langle 2x, 2y, -2z \rangle \quad \nabla h = \langle 1,1,1 \rangle$

$\Rightarrow \; 0 = 2\lambda x + \mu$

$0 = 2\lambda y + \mu$

$1 = -2\lambda z + \mu$

$x^2+y^2 = z^2$

$x+y+z = 24$

$2\lambda x = 2\lambda y \Rightarrow \lambda = 0$ or $x = y$ But $\lambda = 0 \Rightarrow \mu = 0 = 1$

$\Rightarrow$ which is a contradiction so $x = y$.

$2x^2 = z^2$

$\frac{24 \cdot 24}{2}$

$2x + z = 24 \Rightarrow z = (24-2x)$

$\Rightarrow 2x^2 = (24-2x)^2 \Rightarrow 2x^2 = 24^2 + 4x^2 - (2)48x$

$3x^2 - (2)(48)x + (24)^2 = 0 \Rightarrow x = 8$ or 24

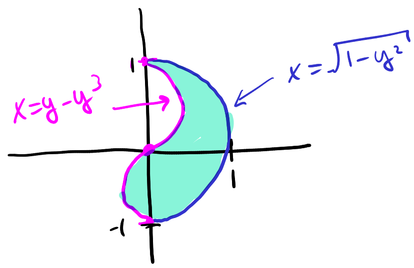
when $x = 8, y = 8 \Rightarrow z = 24 - 16 = 8$

$x = 24, y = 24 \Rightarrow z = 24 - 24 - 24 = -24$

$(8,8,8)$ & $(24,24,-24)$

Problem 6

Evaluate the integral $\int\int_D y dA$ where D is the region in R^2 bounded by $x = y - y^3$ and $x = \sqrt{1-y^2}$



$$\int_{-1}^{1} \int_{y-y^3}^{\sqrt{1-y^2}} y \, dx \, dy$$

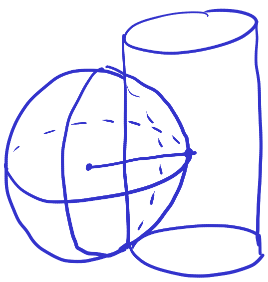
$$= \int_{-1}^{1} y\left(\sqrt{1-y^2} - y + y^3\right) dy = \overbrace{\int_{-1}^{1} y\sqrt{1-y^2}\, dy}^{=A} - \overbrace{\int_{-1}^{1} (y^2 + y^4)\, dy}^{=B}$$

$$A: \begin{cases} u = 1-y^2 \quad du = -2y\,dy \implies -\frac{1}{2} du = y\,dy \\ -\frac{1}{2}\int_{\square}^{\square} u^{1/2} du = \left(-\frac{1}{2}\right)\left(\frac{2(1-y^2)^{3/2}}{3}\right)\Bigg|_{-1}^{1} = 0 \end{cases}$$

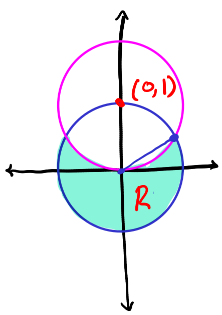
$$B: \frac{y^5}{5} + \frac{y^3}{3}\Bigg|_{-1}^{1} = \frac{2}{5} + \frac{2}{3} = \frac{6+10}{15} = \frac{16}{15}$$

Problem 7

Set up but **do not evaluate** the integral representing the volume of the solid contained **inside** the sphere $1 = x^2 + y^2 + z^2$ and **outside** the cylinder $x^2 + (y-1)^2 = 1$



- This is the same as twice the volume under $z = \sqrt{x^2+y^2-1}$ and over the region R:



- The pink circle has equation $r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta = 0$
$\Rightarrow r^2 - 2r\sin\theta = 0 \Rightarrow r = 2\sin\theta$
- The intersection happens when $\sin\theta = \frac{1}{2} \Rightarrow \theta = \pi/6,\ \theta = \frac{5\pi}{6}$

$$\boxed{\int_{-5\pi/6}^{\pi/6}\int_0^1 \sqrt{r^2-1}\, r\, dr\, d\theta - \int_0^{\pi/6}\int_0^{2\sin\theta} \sqrt{r^2-1}\, r\, dr\, d\theta - \int_{5\pi/6}^{\pi}\int_0^{2\sin\theta} \sqrt{r^2-1}\, r\, dr\, d\theta}$$

Problem 8

Find the surface area of the part of the sphere $4 = x^2 + y^2 + z^2$ that lies above the plane $z = 1$.

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA \qquad z = \sqrt{4-(x^2+y^2)}$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{-x^2-y^2+4}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{-x^2-y^2+4}}$$

$$\boxed{\begin{array}{l}\text{when } z=1 \\ r^2=3\end{array}}$$

Let $x = r\cos\theta$ + $y = r\sin\theta$

then $\frac{\partial z}{\partial x} = \frac{r\cos\theta}{\sqrt{-r^2+4}}$ $\quad \frac{\partial z}{\partial y} = \frac{r\sin\theta}{\sqrt{-r^2+4}}$

$$\Rightarrow 1 + f_x^2 + f_y^2 = \frac{-r^2+4}{-r^2+4} + \frac{r^2\cos^2\theta}{-r^2+4} + \frac{r^2\sin^2\theta}{-r^2+4} = \frac{-r^2+r^2+4}{-r^2+4} = \frac{4}{4-r^2}$$

$$\Rightarrow 2\int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{\frac{1}{4-r^2}}\, r\,dr\,d\theta$$

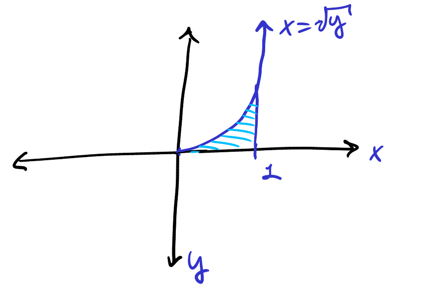
$$\frac{-1}{2} + \frac{2}{2} =$$

Let $u = 4-r^2$ $\quad du = -2r\,dr \Rightarrow -\frac{1}{2}du = r\,dr$

$$\frac{4\pi}{2}\int_1^4 \sqrt{\frac{1}{u}}\, du = 2\pi\int_1^4 u^{-1/2}\,du = 4\pi \cdot u^{1/2}\Big|_1^4 = 4\pi(2-1) = 4\pi$$

Problem 9

Evaluate the integral $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy$



$$\int_0^1 \int_0^{x^2} \frac{y e^{x^2}}{x^3} \, dy \, dx$$

$$= \int_0^1 \frac{e^{x^2}}{x^3} \left(\frac{y^2}{2} \Big|_0^{x^2} \right) dx$$

$$= \int_0^1 \frac{e^{x^2} \cdot x^4}{2x^3} \, dx = \frac{1}{2} \int_0^1 e^{x^2} \cdot x \, dx$$

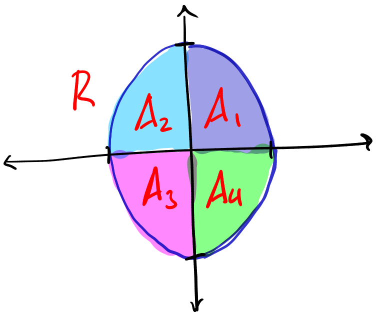
Let $u = x^2$ $\quad du = 2x\,dx \Rightarrow \frac{1}{2} du = x\,dx$

$$\frac{1}{4} \int_0^1 e^u \, du = \frac{e}{4} - \frac{1}{4}$$

Problem 10

Integrate the function $f(x, y) = xy$ over the ellipse $4x^2 + y^2 = 1$.

Note that $f(-x,y) = -f(x,y)$ & $f(x,y) = f(-x,-y)$



$$\iint_R xy\,dA = \underbrace{\iint_{A_1} xy\,dA}_{V_1} + \underbrace{\iint_{A_2} xy\,dA}_{V_2} + \underbrace{\iint_{A_3} xy\,dA}_{V_3} + \underbrace{\iint_{A_4} xy\,dA}_{V_4}$$

But $V_1 = V_3$ & $V_2 = V_4$ and $V_1 = -V_2$

$$\Rightarrow \iint_R xy\,dA = 2V_1 + 2V_2 = 2V_1 - 2V_1 = 0$$