MIDTERM 2

MTH 164 Summer Session B

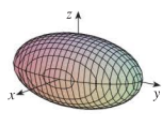
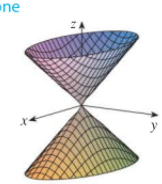
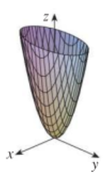
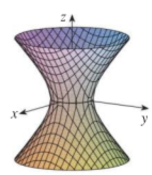
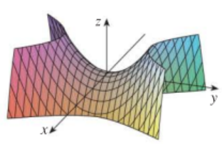
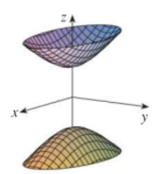
Monday July 25, 2022

Instructions

- **ABSOLUTELY NO COLLABORATION IS ALLOWED ON THIS EXAM**. You are not to communicate in any way with your fellow students during this exam.
- This exam will be proctored over Zoom. **You will get an automatic 0 % on this exam if you take it without being in the Zoom meeting with your camera on**. During the exam, your **face and hands** must be in full view of your camera for the entire duration of the exam. Typing is not allowed during the exam and will be considered suspicious behavior. Once the exam begins, you should only touch your computer to scroll through the pdf so typing should not be necessary.
- If you have questions about the exam, or want to ask to go to the bathroom during the exam, please communicate with me **through chat** so as not to disturb your classmates.
- You must justify all your work completely. No credit will be given to answers without justification unless otherwise stated explicitly in the problem.
- You may NOT look at your textbook, notes, the class notes, or any other resources during the exam. The only resource you should use during the exam is what is provided on the pdf on Gradescope.
- You will write your solutions to the below problems on paper using a pen or pencil. Tablets or other digital writing devices are not allowed.
- After you finish writing up your answers, you may use your phone camera to scan your exam. **Please ask for permision to start scanning via the chat on Zoom before touching your phone**. You will submit your answers as a single pdf file to Gradescope. Once you begin scanning, you will **not be allowed to write anything else on your exam.**
- Absolutely no calculators or calculating websites are allowed on this exam. You will not be required to write out approximate numerical solutions on this exam so please leave your answers in their exact form. For example, if the answer to a question is π, do *not* write $3.14159\ldots$. Just write π.
- You have **1 hour and 15 minutes after the time you recieve the exam pdf document** to complete this exam. After this time period, you will be asked to put your pencil or pen down and begin the scanning process.

Formulas (from Midterm 1)

- Vector equation for a plane: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$
- Vector equation for a line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$
- Vector projection of $\vec{b}$ onto $\vec{a}$: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$
- Scalar projection of $\vec{b}$ onto $\vec{a}$: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$
- Distance between a point and a plane: $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$
- Arc length of a curve $\vec{r}(t)$: $\int_a^b |\vec{r}\,'(t)|dt$
- Classification of quadric surfaces:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

New Formulas

- Polar/cylindrical coordinates: $x = r\cos(\theta)$ $y = r\sin(\theta)$ where $0 \le \theta \le 2\pi$.
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$ $y = \rho\sin(\phi)\sin(\theta)$ $z = \rho\cos(\phi)$ where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$
- The area of the surface with equation $z = f(x, y), (x, y) \in D$ with continuous partial derivatives:

$$A(S) = \int\int_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1} dA$$

- The Jacobian of the transformation T viven by $x = g(u, v)$ and $y = h(u, v)$ is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- The 2nd Derivative test: Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b). Assume that (a, b) is a critical point of f. Define:

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix}.$$

  1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local **maximum** value at (a, b).
  2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local **minimum** value at (a, b).
  3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
  4. If $D(a, b) = 0$, then the test is **inconclusive**.

Formulas from Calc I and II

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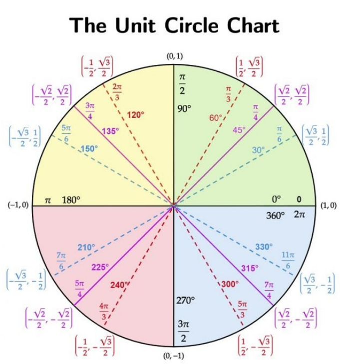

- Integration by parts:

$$\int u dv = uv - \int v du$$

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Problem 1

Let $f(x, y) = \sin(x)\cos(y)$. Find and classify the critical points of $f(x, y)$.

Hint: Recall that the sine and cosine functions are bounded between -1 and 1.

$f_x = \cos(x)\cos(y) = 0 \Rightarrow \cos(x) = 0$ or $\cos(y) = 0$

$f_y = -\sin(x)\sin(y) = 0 \Rightarrow \sin(x) = 0$ or $\sin(y) = 0$

$\cos(x) = 0 \Rightarrow \sin(x) \neq 0 \Rightarrow \sin(y) = 0$

$x = \frac{\pi}{2} + \pi n$ $\qquad$ $y = \pi n$

$\cos(y) = 0 \Rightarrow \sin(y) \neq 0 \Rightarrow \sin(x) = 0$

$y = \frac{\pi}{2} + \pi n$ $\qquad$ $x = \pi n$

$f(\pi/2, 0) = \sin(\pi/2) \cdot \cos(0) = 1$

$f(3\pi/2, 0) = \sin(3\pi/2) \cdot \cos(0) = -1$

$f(0, \pi/2) = \sin(0) \cdot \cos(\pi/2) = 0$

$\therefore$ Local max: $\{(x,y) \in \mathbb{R}^2 : x = \frac{\pi}{2} + 2\pi n$ & $y = 2\pi n$ or $x = \frac{3\pi}{2} + 2\pi n$ & $y = \pi + 2\pi n\}$

Local min: $\{(x,y) \in \mathbb{R}^2 : x = \frac{\pi}{2} + 2\pi n$ & $y = \pi + 2\pi n$ or $x = \frac{3\pi}{2} + 2\pi n$ & $y = \pi n\}$

saddle points: $\{(x,y) \in \mathbb{R}^2 : x = \pi n$ or $y = \frac{\pi}{2} + \pi n\}$

Problem 2

Find the extrema of $f(x, y, z) = xyz$ subject to the constraints

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad z^2 = x^2 + y^2$$

$\nabla f = \langle yz, xz, xy \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$\nabla h = \langle 2x, 2y, -2z \rangle$

System:

$yz = 2x(\lambda + \mu)$

$xz = 2y(\lambda + \mu)$

$xy = 2z(\lambda - \mu)$

$x^2 + y^2 + z^2 = 1$

$z^2 = x^2 + y^2$

$\frac{yz}{x} = \frac{xz}{y} \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$

$2x^2 + z^2 = 1$, $z^2 = 2x^2 \Rightarrow 2z^2 = 1 \Rightarrow z = \pm\sqrt{\frac{1}{2}}$

$\Downarrow$

Note that $x, y, z \neq 0$

$2x^2 = \pm\frac{1}{2} \Rightarrow x = \pm\frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$

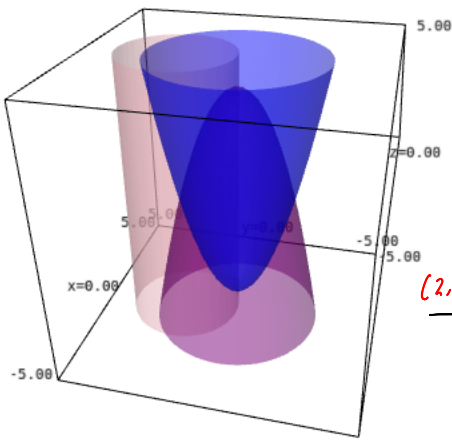
Solutions:

max: $\langle -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$ $\langle -\frac{1}{2}, \frac{1}{2}, \frac{-1}{\sqrt{2}} \rangle$ $\langle \frac{1}{2}, -\frac{1}{2}, \frac{-1}{\sqrt{2}} \rangle$ $\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$

min: $\langle -\frac{1}{2}, -\frac{1}{2}, \frac{-1}{\sqrt{2}} \rangle$ $\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$ $\langle \frac{1}{2}, \frac{1}{2}, \frac{-1}{\sqrt{2}} \rangle$ $\langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$

Problem 3

Set up *but do not evaluate* an integral whose value is equal to the **volume** of the solid that is **below** the paraboloid $z = -x^2 - y^2 + 4$, **above** the paraboloid $z = x^2 + y^2 - 4$, and **inside** the cylinder $x^2 + (y-2)^2 = 4$. A picture of these surfaces is shown below.

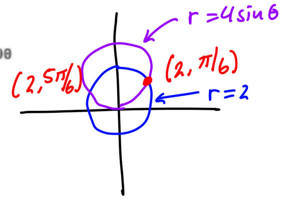


paraboloids intersect at: $x^2+y^2=4 \rightsquigarrow r=2$

cylinder: $r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 = 4$

$\iff r^2 - 4r\sin\theta = 0$

$\iff r = 4\sin\theta$



$$\int_{\pi/6}^{5\pi/6}\int_0^2\int_{r^2-4}^{-r^2+4} r\,dz\,dr\,d\theta + 2\int_0^{\pi/6}\int_0^{4\sin\theta}\int_{r^2-4}^{-r^2+4} r\,dz\,dr\,d\theta$$

Problem 4: Evaluate the integrals

(You may assume that Fubini's theorem applies)

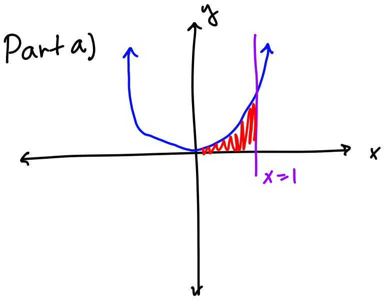
Part a (5 points):

$$\int_0^1 \int_{y^{1/4}}^1 \frac{1}{\sqrt{x^5+1}} dx dy$$

Part b (5 points):

$$\int_1^e \int_{ln(y)}^1 \cos(e^x - x) dx dy$$

Part a)



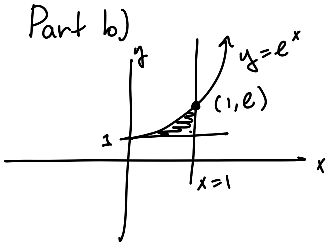
$x = y^{1/4} \quad y = x^4$

$$\int_0^1 \int_0^{x^4} \frac{1}{\sqrt{x^5+1}} dy\, dx = \int_0^1 \frac{x^4}{\sqrt{x^5+1}} dx$$

$u = x^5+1 \quad du = 5x^4 dx \quad \frac{1}{5} du = x^4 dx \quad$ when $x=0, u=1$. when $x=1, u=2$

$$\frac{1}{5}\int_1^2 u^{-1/2} du = \frac{2}{5} u^{1/2} \Big|_1^2 = \frac{2}{5} \cdot (\sqrt{2} - 1)$$

Part b)



$$\int_0^1 \int_1^{e^x} \cos(e^x - x) dy\, dx = \int_0^1 \cos(e^x - x)(e^x - 1) dx$$

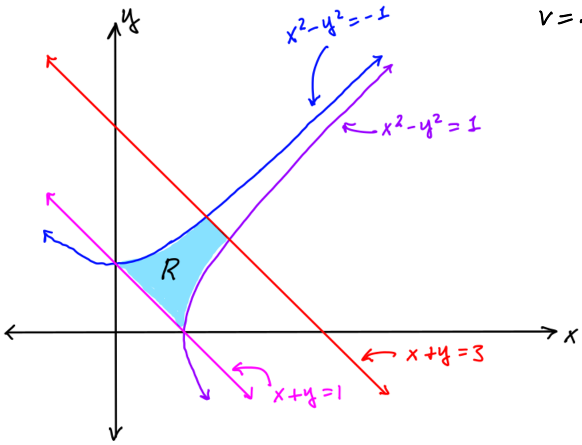
$u = e^x - x \quad du = (e^x - 1) dx \quad$ when $x=0, \; u=1 \quad$ when $x=1, \; u = e-1$

$$\int_1^{e-1} \cos(u) du = \sin(e-1) - \sin(1)$$

Problem 5

Part a: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by the equations $u = x - y$ and $v = x + y$. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

Part b: Evaluate $\int\int_R (x-y)e^{x^2-y^2}\,dA$ where R (pictured below) is bounded by the lines $x+y = 1$, $x+y = 3$, and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$.



$u = x-y$ $\quad u+v = 2x \quad x = \frac{u+v}{2} = \frac{1}{2}u + \frac{1}{2}v$

$v = x+y$ $\quad u-v = -2y \quad y = \frac{u-v}{-2} = -\frac{1}{2}u + \frac{1}{2}v$

$$\begin{vmatrix} \partial x/\partial u = \frac{1}{2} & \frac{\partial x}{\partial v} = \frac{1}{2} \\ \partial y/\partial u = -\frac{1}{2} & \partial y/\partial v = \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

$x = \frac{u+v}{2}$ $\qquad x^2 - y^2 = -1 \rightsquigarrow \frac{1}{4}(u^2+2uv+v^2) - \frac{1}{4}(u^2-2uv+v^2) = -1$

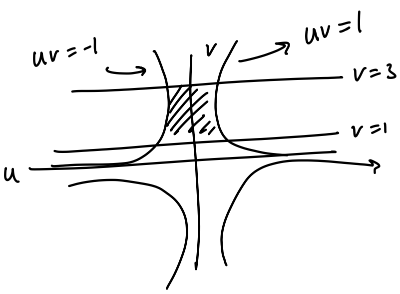
$y = \frac{u-v}{-2}$ $\qquad \Rightarrow u^2-u^2+4uv+v^2-v^2 = -4 \Rightarrow uv = -1$

Similarly $\quad x^2+y^2 \rightsquigarrow uv = 1$

$x+y=3 \Rightarrow y = 3-x \rightsquigarrow v = x+3-x = 3$ $\qquad$ 12:29

$x+y=1 \Rightarrow y = 1-x \rightsquigarrow v = 1$

$u = x-y$
$v = x+y$



$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} u e^{uv}\,du\,dv$$

Integration by parts:

$dv = e^{ax} \Rightarrow v = \frac{1}{a}e^{ax}$ $\qquad \int u\,dv = u\cdot v - \int v\,du$

$u = x \Rightarrow dv = 1\,dx$ $\qquad = \frac{x}{a}\cdot e^{ax} - \frac{1}{a}\int e^{ax}dx = \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + C$

$$\frac{1}{2}\int_1^3 \left(\frac{u}{v}\cdot e^{uv} - \frac{1}{v^2}e^{uv}\right)\Big|_{-1/v}^{1/v} dv = \frac{1}{2}\int_1^3\left(\left(\frac{1}{v^2}\cdot e - \frac{1}{v^2}\cdot e\right) - \left(\frac{-1}{v^2}\cdot e^{-1} - \frac{1}{v^2}e^{-1}\right)\right)dv$$

$$= \int_1^3 \left(\frac{1}{v^2\cdot e}\right)dv = \frac{1}{ve}\Big|_1^3 = \frac{1}{3e} - \frac{1}{e} = \frac{1}{3e} - \frac{3}{3e} = -\frac{2}{3e}$$