MIDTERM 1

MTH 164 Summer Session B

Monday July 13, 2020

Instructions

- **ABSOLUTELY NO COLLABORATION IS ALLOWED ON THIS EXAM**. You are not to communicate in any way with your fellow students during this exam.
- You must justify all your work completely. No credit will be given to answers without justification.
- You may look at your textbook and notes during the exam. You may use theorems from the textbook without proof but you must prove results stated in exercises or examples if you use them.
- You will write your solutions to the below problems on paper, or type them using Latex. If you are writing your solutions on paper, please scan them using a phone app or scanner. Write one problem per page and clearly label which problem goes with what work.
- You have **1 hour and 15 minutes after the time you recieve the exam pdf document** to complete this exam. You will have 15 minutes to then send me your solutions in an email, or upload your solutions to Blackboard. The total time from when you recieve the exam to when I recieve your completed work is one hour and thirty minutes.
- Please send your solutions in pdf format. Pictures will be accepted in an emergency if your scanning methods fail you. **Put your name in the filename**. Send your solutions in the order they appear in this exam.

Problem 1

a) Prove the identity:

$$(3\vec{u} + 2\vec{v}) \times (4\vec{u} + 2\vec{v}) = 2\vec{v} \times \vec{u}$$

b) Let $\vec{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\vec{w} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Find a scalar s such that $\vec{v}$ is orthogonal to $\vec{v} - s\vec{w}$

a) $(3u + 2v) \times (4u + 2v) = (3u + 2v) \times 4u + (3u + 2v) \times 2v$

$= \underbrace{3u \times 4u}_{0} + 2v \times 4u + 3u \times 2v + \underbrace{2v \times 2v}_{0}$

$= 8(v \times u) + 6(u \times v)$

$= 8(v \times u) - 6(v \times u) = 2 v \times u$ as desired.

b) Let $v = \langle 2, -1, 3 \rangle$, $w = \langle 3, -1, 4 \rangle$

If $v \perp (v - sw)$, then $v \cdot (v - sw) = 0$

We have $v \cdot (v - sw) = v \cdot v - v \cdot sw$

so $v \cdot (v - sw) = 0 \implies |v|^2 = v \cdot sw$

$|v|^2 = 4 + 1 + 9 = 14$

$v \cdot sw = s(2)(3) + s(-1)(-1) + s(3)(4)$

$= s(6 + 1 + 12)$

$= 19s$

$\implies 14 = 19s \implies \boxed{s = \frac{14}{19}}$

Problem 2

Let $\vec{r}(t) = \langle 2t+1, t, 2-3t \rangle$ describe the path of a particle, where the parameter t represents time. Let P be the plane $0 = (x-1) + 2(z-2)$. Let $D(t)$ be the distance between the particle and the plane P at time t. How fast is $D(t)$ changing at $t = 0$? Is the particle approaching or going away from the plane at $t = 0$?

The distance from a point (x_0, y_0, z_0) to a plane $ax + by + c + d = 0$:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

When $t=0$, the particle is at the point $\vec{r}(0) = \langle 1, 0, 2 \rangle$

Rearranging the plane equation: $x + 2z - 1 - 4 = x + 2z - 5 = 0$

So distance as a function of time is

$$D(t) = \frac{|2t+1 + 2(2-3t)|}{\sqrt{5}} = \frac{|2t - 6t + 1 + 4|}{\sqrt{5}} = \frac{|-4t+5|}{\sqrt{5}}$$

Thus $D'(t) = \frac{|-4|}{\sqrt{5}} = \frac{4}{\sqrt{5}}$ including $t=0$.

Note that $D(t) = 0$ when $-4t + 5 = 0 \Rightarrow -4t = -5 \Rightarrow t = \frac{5}{4}$

So the particle hits the plane at $t = \frac{5}{4}$.

$0 < \frac{5}{4} \Rightarrow$ the particle is approaching the plane when $t=0$.

Problem 3

Let P_1 bet the plane $2x - 3y + z - 4 = 0$ and P_2 be the plane $3x + y + 4z = 0$. Find the equation of the plane P_3 which passes through the origin and is perpendicular to both P_1 and P_2.

$$n = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 3 & 1 & 4 \end{vmatrix} = \langle -13, -5, 11 \rangle$$

Plane equation $\langle r - r_0 \rangle \cdot n = 0$

$\vec{r}_0 = (0,0,0)$

$$\langle x, y, z \rangle \cdot \langle -13, -5, 11 \rangle = 0$$

$$\boxed{-13x - 5y + 11z = 0}$$

Problem 4

Find the paramtetric equations representing the curve (or curves) of intersection of the surface $\frac{x^2}{9}+y^2+\frac{z^2}{9}=1$ and the plane $x=z$.

Since $x=z$, $\frac{2t^2}{9}+y^2=1$ which is the equation of an ellipse.

the parametric equations are
$$\boxed{\begin{aligned} x &= \frac{3}{\sqrt{2}}\cos(t) \\ y &= \sin(t) \\ z &= \frac{3}{\sqrt{2}}\cos(t) \end{aligned}}$$

$$\boxed{\vec{r}(t) = \left\langle \frac{3}{\sqrt{2}}\cos(t),\ \sin(t),\ \frac{3}{\sqrt{2}}\cos(t) \right\rangle}$$

Problem 5

A spaceship is travelling along the curve defined by $\vec{r}(t) = \langle t, 2cos(t), 2sin(t)\rangle$ at a speed of 2π units per hour. A spaceport is located at the point $A(0, 2, 0)$. A fuel station is located at the point $B(4\pi, 2, 0)$. Approximately how long does it take the spaceship to travel from point A to point B?

Warning: Note that in this question, the parameter t does not represent time.

Distance from A to B:

$$L = \int_0^{4\pi} \sqrt{1+4}\,dt = \sqrt{5}\cdot 4\pi$$

$$\frac{2\pi \text{ units}}{1 \cdot \text{hour}} = \frac{\sqrt{5}\,4\pi \text{ units}}{x \text{ hours}} \Rightarrow \sqrt{5}\,4\pi = 2\pi x$$

$$\Rightarrow x = 2\sqrt{5} \text{ hours}$$

Problem 6

Let $f_c(x) = e^{cx}$. For what values of c does $f_c(x)$ have the largest curvature at $x = 0$?

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

$$f_c'(x) = c e^{cx} \qquad f_c''(x) = c^2 e^{cx}$$

$$\Rightarrow \kappa_c(x) = \frac{c^2 e^{cx}}{[1 + c^2 e^{2cx}]^{3/2}}$$

$$\kappa_c(0) = \frac{c^2}{(1+c^2)^{3/2}}$$

Calculate $\frac{d\kappa_c(0)}{dc}$ & set $= 0$. Solve for c.

$$\boxed{c = \pm\sqrt{2}}$$

Problem 7

a) Find the domain and range of $f(x, y) = cos(x) + sin(y)$

b) Find the set of points in $\mathbb{R}^2$ where $f(x, y) = \frac{1}{4-x^2-y^2}$ is continuous.

a) D: All of $\mathbb{R}^2$

R: $[-2, 2]$

b) $\mathbb{R}^2 - c$ where $c = \{(x,y) : x^2+y^2=1\}$

Problem 8

A particle is moving with position function $\vec{r}(t)$ (here the parameter t does represent time). Show that if $|\vec{r}\,'(t)| = C$ where C is a constant, then the velocity and acceleration vectors are orthogonal.

$$\frac{d}{dt}\left(r'(t)\cdot r'(t)\right) = r'(t)\cdot r''(t) + r''(t)\cdot r'(t) = 2r'(t)\cdot r''(t) = 0$$

since $\left(r'(t)\cdot r(t)\right) = |\vec{r}\,'(t)|^2 = C^2$

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Problem 9

Let $L_1 = \langle 2{+}3t, 5{-}t, 3t\rangle$ and $L_2 = \langle t, 4t, 2t\rangle$. Determine whether L_1 and L_2 are parallel, skew, or intersecting. If they intersect, determine their intersection point.

$\langle 3,-1,3\rangle \neq \lambda\langle 1,4,2\rangle \Rightarrow$ not parallel

$2+3t = s$

$5-t = 4s$

$3t = 2s$

$t = \frac{2}{3}s$

$2+\frac{2}{3}\cdot 3 = s \Rightarrow \boxed{s=4}$ $\boxed{t=\frac{8}{3}}$

$5-\frac{8}{3} = \frac{15-8}{3} = \frac{7}{3} \neq 16 \Rightarrow$ they do <u>not</u> intersect.

L_1 + L_2 are <u>skew</u>

Problem 10

a) Is the vector function $\vec{r}(t) = \langle t^2 sin(t), t^2, t^2 cos(t)\rangle$ a smooth parameterization? Why or why not?

b) For what values of t does $\vec{r}(t)$ have double points? (A curve $\vec{\alpha}(t)$ has a **double point** at t_0 if there exists t_1 such that $\vec{\alpha}(t_0) = \vec{\alpha}(t_1)$).

a) $\underline{\text{No}}$ since $r'(0) = 0$

b) • $t_1^2 = t_2^2 \iff t_1 = \pm t_2$

• $\sin(t) = \sin(-t) \iff x = \pi \cdot n, \; n \in \mathbb{Z}$

• $\cos(t) = \cos(-t) \;\; \forall t$

$\Rightarrow \vec{r}(t)$ has double points at $t = \pi \cdot n \;\; \forall n \in \mathbb{Z}$