Solutions

MIDTERM 1

MATH 164 Summer Session B

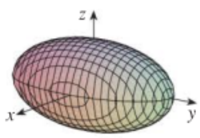
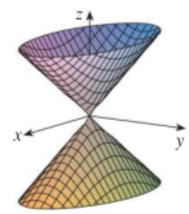
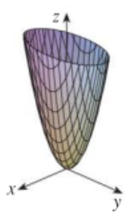
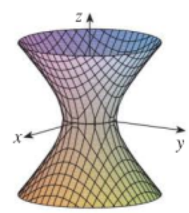
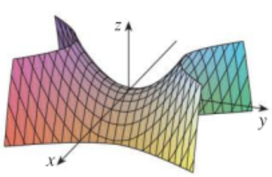
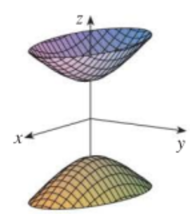
Monday July 11, 2022

Instructions

- **ABSOLUTELY NO COLLABORATION IS ALLOWED ON THIS EXAM**. You are not to communicate in any way with your fellow students during this exam.
- This exam will be proctored over Zoom. **You will get an automatic 0 % on this exam if you take it without being in the Zoom meeting with your camera on**. During the exam, your **face and hands** must be in full view of your camera for the entire duration of the exam. Typing is not allowed during the exam and will be considered suspicious behavior. Once the exam begins, you should only touch your computer to scroll through the pdf so typing should not be necessary.
- If you have questions about the exam, or want to ask to go to the bathroom during the exam, please communicate with me **through chat** so as not to disturb your classmates.
- You must justify all your work completely. No credit will be given to answers without justification unless otherwise stated explicitly in the problem.
- You may NOT look at your textbook, notes, the class notes, or any other resources during the exam. The only resource you should use during the exam is what is provided on the pdf on Gradescope.
- You will write your solutions to the below problems on paper using a pen or pencil. Tablets or other digital writing devices are not allowed.
- After you finish writing up your answers, you may use your phone camera to scan your exam. **Please ask for permision to start scanning via the chat on Zoom before touching your phone**. You will submit your answers as a single pdf file to Gradescope. Once you begin scanning, you will **not be allowed to write anything else on your exam.**
- Absolutely no calculators or calculating websites are allowed on this exam. You will not be required to write out approximate numerical solutions on this exam so please leave your answers in their exact form. For example, if the answer to a question is π, do *not* write $3.14159\ldots$. Just write π.
- You have **1 hour and 15 minutes after the time you recieve the exam pdf document** to complete this exam. After this time period, you will be asked to put your pencil or pen down and begin the scanning process.

Formulas

- Vector equation for a plane: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$
- Vector equation for a line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$
- Vector projection of $\vec{b}$ onto $\vec{a}$: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$
- Scalar projection of $\vec{b}$ onto $\vec{a}$: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$
- Distance between a point and a plane: $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$
- Arc length of a curve $\vec{r}(t)$: $\int_a^b |\vec{r}\,'(t)|dt$
- Classification of quadric surfaces:

Surface	Equation	Surface	Equation
Ellipsoid  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone  	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets  	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Problem 1

Find a parametric equation for the curve (or curves) of intersection between the surface $-x^2 - \frac{y^2}{4} + z^2 = 1$ and the surface $x^2 + \frac{y^2}{4} - 3z - 3 = 0$. If the "curve" is just a point, or set of points, you may write your answer in set notation.

$$(1)\quad -x - \frac{y^2}{4} + z^2 = 1 \implies z^2 - 1 = \boxed{x^2 + \frac{y^2}{4}}$$

$$(2)\quad x^2 + \frac{y^2}{4} - 3z - 3 = 0 \implies 3z + 3 = \boxed{x^2 + \frac{y^2}{4}}$$

$$\implies z^2 - 1 = 3z + 3$$

$$\Downarrow$$

$$z^2 - 3z - 4 = 0$$

Factor:

$$z^2 - 3z - 4 = 0 \implies z = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm \sqrt{25}}{2} = \frac{8}{2} \text{ or } \frac{-2}{2} = 4 \text{ or } -1$$

Case 1: $z = 4$

$$(1) \implies 16 - 1 = x^2 + \frac{y^2}{4} \implies 15 = x^2 + \frac{y^2}{4} \implies 1 = \frac{x^2}{15} + \frac{y^2}{15 \cdot 4}$$

Let $x = \sqrt{15}\cos(t)$, $y = 2\sqrt{15}\sin(t)$.

Note that in this case, $\frac{x^2}{15} + \frac{y^2}{15 \cdot 4} = \frac{15\cos^2 t}{15} + \frac{4 \cdot 15 \sin^2(t)}{4 \cdot 15} = 1$

$\therefore$ One curve of intersection is

$$\boxed{\vec{r}_1(t) = \langle \sqrt{15}\cos(t), 2\sqrt{15}\sin(t), 4 \rangle}$$

Case 2: $z = -1$

$$(1) \implies 1 - 1 = x^2 + \frac{y^2}{4} = 0 \iff x = 0 \text{ and } y = 0$$

$\implies$ The set $\boxed{\{(0,0,-1)\}}$ is also contained in the intersection.

Problem 2

a) Let $l_1(t) = \langle t+1, t-4, 2t\rangle$ and $l_2(s) = \langle s, 1, s+2\rangle$ be two lines. Show that l_1 and l_2 are skew.

b) What is the distance between l_1 and l_2?

a) Try to solve: $\quad t+1,\ t-4,\ 2t$

$t+1 = s$

$t-4 = 1 \Rightarrow t = 5 \Rightarrow s = 6 \Rightarrow 10 = 8$ contradiction. Thus l_1 & l_2 do not intersect.

$2t = s+2$

Furthermore, the direction vector of l_1 is $\vec{v}_1 = \langle 1,1,2\rangle$

while the direction vector of l_2 is $\vec{v}_2 = \langle 1,0,1\rangle$

So l_1 & l_2 are clearly not parallel. Thus, they are skew.

b) $$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = \vec{i}(1-0) - \vec{j}(1-2) + \vec{k}(0-1) = \langle 1, 1, -1\rangle$$

A point on l_2 is $l_2(0) = (0, 1, 2)$

$\Rightarrow$ A plane containing l_2 & parallel to l_1 is given by

$$\langle 1,1,-1\rangle \cdot \langle x, y-1, z-2\rangle = x+y-1-z+2 = x+y-z+1 = 0$$

A point on l_1 is $l_1(0) = (1,-4,0)$

distance from plane to $l_1(0)$: $\dfrac{|1-4+1|}{\sqrt{1+1+1}} = \boxed{\dfrac{2}{\sqrt{3}}}$

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Problem 3

Let $\vec{v} = \langle 1, 2, 1\rangle$ and $\vec{b} = \langle 1, 2, 3\rangle$. Find a vector $\vec{u}$ that is orthogonal to $\vec{v}$ but coplanar with $\vec{v}$ and $\vec{b}$.

Let $\vec{u} = \langle a, b, c\rangle$

- $\vec{u} \perp \vec{v} \Rightarrow \vec{u} \cdot \vec{v} = a + 2b + c = 0$ (1)
- $\vec{u}$ coplanar with $\vec{v}$ & $\vec{b} \Rightarrow \vec{u} \cdot (\vec{v} \times \vec{b}) = 0$

$$\vec{v} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \vec{i}(6-2) - \vec{j}(3-1) + \vec{k}(2-2) = \langle 4, -2, 0\rangle$$

$$\Rightarrow \vec{u} \cdot (\vec{v} \times \vec{b}) = \langle a, b, c\rangle \cdot \langle 4, -2, 0\rangle$$

$$= 4a - 2b = 0 \Rightarrow 2a = b \quad (2)$$

Let $b = 2$, $a = 1$, & plug into (1):

$$1 + 4 + c = 0 \quad \Rightarrow \quad c = -5$$

one solution is: $\boxed{\langle 1, 2, -5\rangle}$

Problem 4

a) Show that the limit exists, and compute the limit.

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) ln(x^2 + y^2)$$

b) Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$$

L'Hopital

a) $\lim_{r\to 0^+} r\ln(r) = \lim_{r\to 0^+} \frac{\ln(r)}{1/r} = \lim_{r\to 0^+} \frac{-1/r}{1/r^2} = \lim_{r\to 0^+} -x = \boxed{0}$

b) when $y=x$: $\lim_{x\to 0} \frac{x^5}{x^2+x^8} = 0$

when $x=y^4$: $\lim_{x\to 0} \frac{x^2}{2x^2} = 1/2$

So since $0 \neq 1/2$, $\boxed{DNE}$

Problem 5

a) Find the arc length of the curve $\vec{r}(t) = \langle 1, t^2, t^3 \rangle$, for $0 \le t \le 1$.

b) Write an equation for the line tangent to the curve when $t = 1$.

a) $\vec{r}\,'(t) = \langle 0, 2t, 3t^2 \rangle$

$\frac{1}{27}\left(13\sqrt{13} - 8\right)$

$|\vec{r}\,'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$

$\Rightarrow L = \int_0^1 t\sqrt{4 + 9t^2}\,dt$

Let $u = 4 + 9t^2$

$du = 18t\,dt \Rightarrow t\,dt = \frac{1}{18}\,du$

when $t = 0$, $u = 4$ when $t = 1$, $u = 13$

$\Rightarrow L = \frac{1}{18}\int_4^{13} u^{1/2}\,du = \frac{1}{18} \cdot \frac{2}{3} u^{3/2}\Big|_4^{13} = \frac{1}{27}\left(13\sqrt{13} - 4\sqrt{4}\right) = \boxed{\frac{1}{27}\left(13\sqrt{13} - 8\right)}$

b) $\vec{v} = \vec{r}\,'(1) = \langle 0, 2, 3 \rangle$

$\vec{r}_0 = \vec{r}(1) = \langle 1, 1, 1 \rangle$

$\Rightarrow \vec{r}(t) = \langle 1, 1, 1 \rangle + \langle 0, 2t, 3t \rangle$

$\boxed{= \langle 1, 1+2t, 1+3t \rangle}$

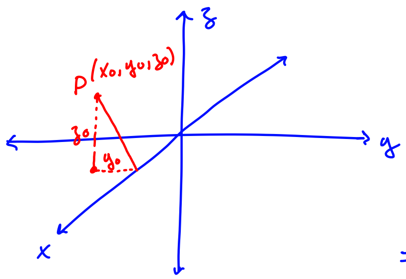
Problem 6

Find an equation for the quadric surface consisting of all the points P for which the distance from P to the x-axis is twice the distance from P to the yz plane. Identify the surface.

Let $P = \langle x, y, z \rangle$

the x-axis is $\vec{r}(x) = \langle x, 0, 0 \rangle$

The yz-plane is $\langle 1,0,0 \rangle \cdot \langle x,y,z \rangle = 0 \Rightarrow x = 0$



distance from P to x-axis: $\sqrt{y^2 + z^2}$

distance from P to yz-plane: $|x|$

$\Rightarrow$ The surface is $S = \{ (x,y,z) : \sqrt{y^2+z^2} = 2|x| \}$

or in other words: $S = \{ (x,y,z) : y^2 + z^2 = 4x^2 \}$

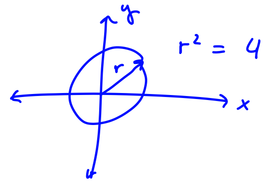
$$\boxed{S = \left\{ (x,y,z) : \frac{y^2}{4} + \frac{z^2}{4} = x^2 \right\}}$$

This is a cone.

Problem 7

Part a) Particle 1 travels in a circle in the xz-plane that has radius 2 and is centered at the origin, with speed = 4 units/second. Write a parametric equation for the particle, where the parameter t represents time.

Part b) Particle 2 travels along the positive y-axis, starting at the origin, with speed equal to $2t$. How fast is particle 2 moving away from particle 1 at time t?



$r^2 = 4 \iff r = 2$

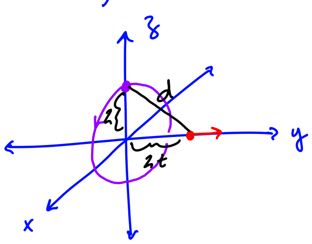
a) Circle w/ radius 2: $\vec{r}(t) = \langle 2\cos At, 2\sin At \rangle$

Velocity: $\vec{r}\,'(t) = \langle -2A\sin(At), 2A\cos(At) \rangle$

Speed: $|\vec{r}\,'(t)| = \sqrt{4A^2} = 2A = 4 \iff A = \pm 2$

$$\boxed{\vec{r}(t) = \langle 2\cos(2t), 2\sin(2t) \rangle}$$

b)



$d(t) = \sqrt{4 + 4t^2}$

$$d'(t) = \frac{1}{2} \cdot 8t\,(4 + 4t^2)^{-1/2} = \frac{4t}{\sqrt{4+4t^2}} = \boxed{\frac{2t}{\sqrt{1+2t^2}}}$$