Solutions

FINAL EXAM: PART A

MTH 164 Summer Session A

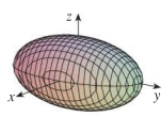
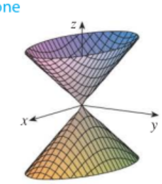
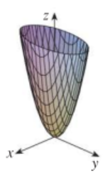
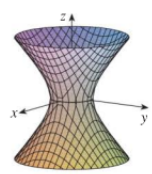
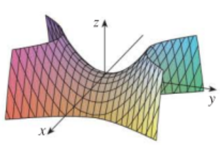
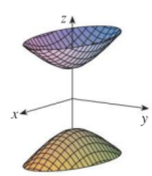
Monday July 1, 2021

Instructions

- **ABSOLUTELY NO COLLABORATION IS ALLOWED ON THIS EXAM**. You are not to communicate in any way with your fellow students during this exam.
- This exam will be proctored over Zoom. **You will get an automatic 0 % on this exam if you take it without being in the Zoom meeting with your camera on**. During the exam, your **face and hands** must be in full view of your camera for the entire duration of the exam. Typing is not allowed during the exam and will be considered suspicious behavior. Once the exam begins, you should only touch your computer to scroll through the pdf so typing should not be necessary.
- If you have questions about the exam, or want to ask to go to the bathroom during the exam, please communicate with me **through chat** so as not to disturb your classmates.
- You must justify all your work completely. No credit will be given to answers without justification unless otherwise stated explicitly in the problem.
- You may NOT look at your textbook, notes, the class notes, or any other resources during the exam. The only resource you should use during the exam is what is provided on the pdf on Gradescope.
- You will write your solutions to the below problems on paper using a pen or pencil. Tablets or other digital writing devices are not allowed.
- After you finish writing up your answers, you may use your phone camera to scan your exam. **Please ask for permision to start scanning via the chat on Zoom before touching your phone**. You will submit your answers as a single pdf file to Gradescope. Once you begin scanning, you will **not be allowed to write anything else on your exam.**
- Absolutely no calculators or calculating websites are allowed on this exam. You will not be required to write out approximate numerical solutions on this exam so please leave your answers in their exact form. For example, if the answer to a question is π, do *not* write $3.14159\ldots$. Just write π.
- You have **1 hour and 30 minutes after the time you recieve the exam pdf document** to complete this exam. After this time period, you will be asked to put your pencil or pen down and begin the scanning process.

Formulas (from Midterm 1)

- Vector equation for a plane: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$
- Vector equation for a line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$
- Vector projection of $\vec{b}$ onto $\vec{a}$: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$
- Scalar projection of $\vec{b}$ onto $\vec{a}$: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$
- Distance between a point and a plane: $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$
- Arc length of a curve $\vec{r}(t)$: $\int_a^b |\vec{r}'(t)|dt$
- Classification of quadric surfaces:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Formulas (from Midterm 2)

- Polar/cylindrical coordinates: $x = r\cos(\theta)$ $y = r\sin(\theta)$ where $0 \leq \theta \leq 2\pi$.
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$ $y = \rho\sin(\phi)\sin(\theta)$ $z = \rho\cos(\phi)$ where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$
- The area of the surface with equation $z = f(x, y), (x, y) \in D$ with continuous partial derivatives:

$$A(S) = \int\int_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}dA$$

- The Jacobian of the transformation T viven by $x = g(u, v)$ and $y = h(u, v)$ is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

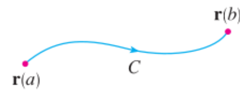
Formulas from Chapter 16

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

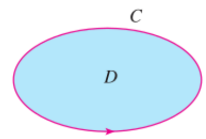
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



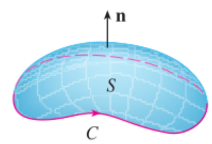
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



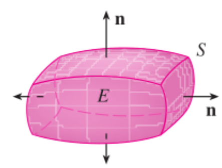
Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



- Positvely oriented curves are **counter-clockwise**.
- The line integral of a function over a curve C given by the vector equation $\mathbf{r}(t)$ for $a \leq t \leq b$ is $\int_C f ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(r)(t)|dt$.
- For a vector field $\mathbf{F}$, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$.
- A vector field $\mathbf{F}$ is called **conservative** if an only if there exists a function f satisfying $\nabla f = \mathbf{F}$. Conservative vector fields are path-independent.
- The area of a domain D enclosed by a curve C is: $A = \frac{1}{2} \oint_C xdy - ydx$.
- $\text{curl}\mathbf{F} = \vec{\nabla} \times \mathbf{F}$. If $\mathbf{F}$ is conservative, it has zero curl.
- $\text{div}\mathbf{F} = \vec{\nabla} \cdot \mathbf{F}$.
- $\text{div curl}\mathbf{F} = 0$

- The area of a parameterized surface $\mathbf{r}(u, v)$ is $A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$
- Surface integral of a function: $\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$
- Surface integral of a vector field: $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$.

Problem 1

Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = \left\langle y\sin(z)e^z, x\sin(z)e^z, xye^z\left(\cos(z)+\sin(z)\right)\right\rangle$$

and C is given by

$$\mathbf{r}(t) = \left\langle (2+\sin(t))\cos(t), (2+\sin(t))\sin(t), \frac{t}{\pi}+\cos(t)\right\rangle \qquad 0 \le t \le 5\pi$$

Hint: Apply the Fundamental Theorem of Line Integrals.

Claim: $\vec{F}$ is conservative.

$f_x(x,y,z) = y\sin z\, e^z \rightsquigarrow f(x,y,z) = xy\sin z\, e^z + k_1(y,z)$

$f_y(x,y,z) = x\sin z\, e^z \rightsquigarrow f(x,y,z) = xy\sin z\, e^z + k_2(x,z)$

$f_z(x,y,z) = xye^z\cos z + xye^z\sin z \rightsquigarrow f(x,y,z) = xy\sin z\, e^z + k_3(x,z)$

$\Rightarrow k_1 = k_2 = k_3 = k \quad \& \quad f(x,y,z) = xy\sin z\, e^z + k \qquad \therefore \vec{F}$ is conservative!

$\vec{r}(0) = \langle 2, 0, 1\rangle$

$\vec{r}(5\pi) = \langle 0, 3, 5\rangle$

$\Rightarrow \int_C \vec{F}\cdot dr = f(0,3,5) - f(2,0,1) = 0 - 0 = 0$!

Problem 2

Find the positively oriented, simple, closed curve C (in the $xy-$plane) that maximizes the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \left\langle y^5 + 2y^2, -x^3 + 4yx + 9x \right\rangle.$$

Hint: Recall that $\mathbf{F} \cdot d\mathbf{r} = Pdx + Qdy$ for vector fields of the form $\mathbf{F} = \langle P, Q\rangle$. This is a Green's theorem problem.

$$\vec{F}\cdot d\vec{r} = \underbrace{(y^5+2y^2)}_{P}dx + \underbrace{(-x^3+4yx+9x)}_{Q}dy$$

$$\frac{\partial Q}{\partial x} = -3x^2+4y+9$$

$$\frac{\partial P}{\partial y} = 5y^4+4y$$

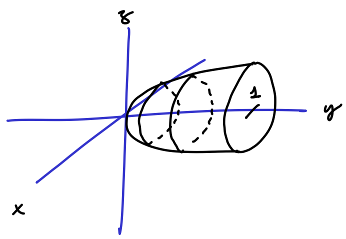
$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -3x^2+4y+9-5y^4-4y = -3x^2-5y^4+9$$

Note that $-3x^2-5y^4$ get large as $x, y \to \infty$. $\Rightarrow$ the equation $f(x,y)$ is positive inside the curve defined by $-3x^2-5y^4+9=0$ + negative outside it.

$\Rightarrow$ C is given by $3x^2+5y^4=9$

Problem 3

Evaluate the surface integral: $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = \langle 0, y, -z \rangle$ and S is the part of the paraboloid $y = x^2 + z^2$ with $0 \le y \le 1$. **Hint:** There are several ways to do this problem. Recall that $\int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \cos^2 dx = \pi$. This might be useful depending on which method you choose.



$f(x,y,z) = x^2 + z^2 - y$

$\nabla f = \langle 2x, -1, 2z \rangle$

$\int_0^{2\pi} \int_0^1 \langle 0, x^2+z^2, -z \rangle \cdot \langle 2x, -1, 2z \rangle \, dA$

$= \int_0^{2\pi} \int_0^1 (-x^2 - z^2 - 2z^2) \, r \, dr \, d\theta$ $\quad x = r\cos\theta$, $y = r\sin\theta$

S^1

$= \int_0^{2\pi} \int_0^1 (-r^3\cos^2\theta - 3r^3\sin^2\theta) dr \, d\theta$

$= \int_0^{2\pi} \left(-\frac{1}{4}\cos^2\theta - \frac{3}{4}\sin^2\theta\right) d\theta = -\frac{1}{4}\pi - \frac{3}{4}\pi = -\pi$

S^2 $\quad \int_0^{2\pi} \int_0^1 \langle 0, 1, -z \rangle \cdot \langle 0, 1, 0 \rangle \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta$

$= 2\pi \cdot \frac{1}{2} = \pi$

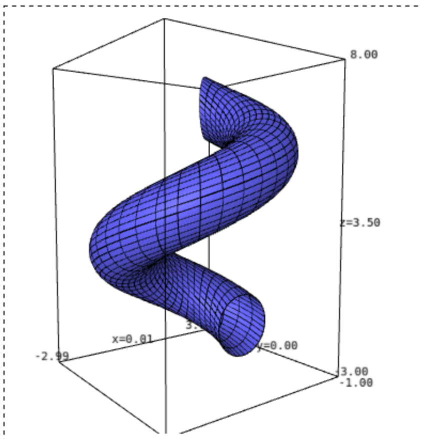
Problem 4

Part a (3 points): Let $\mathbf{F} = \langle 0, 0, 1\rangle$. Find a vector field $\mathbf{G}$ that satisfies $\vec{\nabla} \times \mathbf{G} = \mathbf{F} = \langle 0, 0, 1\rangle$.

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ a & b & c \end{vmatrix} = \langle \partial_y c - \partial_z b, -\partial_x c + \partial_z a, \partial_x b - \partial_y a \rangle$$

$$\vec{F} = \langle a, b, c\rangle = \langle 0, x, 0\rangle$$

Part b (7 points): Recall the noodle shape whose parametric equations are $\mathbf{r}(u, v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u+\cos v\rangle$. Calculate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is the portion of the noodle with $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ and $\mathbf{F}(x, y, z) = \langle 0, 0, 1\rangle$. **Hint:** You might want to apply Stokes' theorem.



$$\iint_S \vec{\nabla}\times G = \int_{\partial S} G$$

Let C_1 be the curve at $u=0$:

$$\vec{r}_1(v) = \langle 2+\sin v, 2+\sin v, \cos v\rangle$$

Let C_2 be the curve at $u=0$:

$$\vec{r}_2(v) = \langle 2+\sin v, 2+\sin v, 2\pi+\cos v\rangle$$

$$\vec{r}_1{}'(v) = \langle \cos v, \cos v, -\sin v\rangle = \vec{r}_2{}'(v)$$

$$\int_{C_2} \vec{G}\cdot d\vec{r} = \int_{C_1} \vec{G}\cdot d\vec{r} = \int_0^{2\pi} \langle 0,0,1\rangle \cdot \langle \cos v, \cos v, -\sin v\rangle\, dv$$

$$= \int_0^{2\pi} (-\sin v)\, dv = \cos v \Big|_0^{2\pi} = 0$$

$\Rightarrow$ By Stokes' theorem, $\iint_S \vec{F}\cdot d\vec{S} = 0$

Problem 5

Calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = \langle x^2yz, xy^2z, xyz^2 \rangle$ and S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$ where a, b, and c are positive numbers.
Hint: Use the Divergence Theorem.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$6 \int_0^c \int_0^b \int_0^a xyz \, dx \, dy \, dz$$

$$= 6 \left. \frac{x^2}{2} \right|_0^a \cdot \left. \frac{y^2}{2} \right|_0^b \cdot \left. \frac{z^2}{2} \right|_0^b = \frac{6}{8} a^2 b^2 c^2 = \frac{3}{4} a^2 b^2 c^2$$