Wednesday, August 3, 2022

MATH 164 Lecture Notes

Last Review Day!

Lagrange Multipliers

1. Find the max/min of $f(x,y) = 18x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$

2. Find the max/min of $f(x,y,z) = 3x^2 + y$ subject to the constraints $4x - 3y = 9$ & $x^2 + z^2 = 9$

Absolute Max/Min

1. Find the absolute max/min of $f(x,y) = x^2 + y^2 + 4x - 6y$ on the domain $x^2 + y^2 \le 16$

Critical Points

→ 1. Find the point (x,y) of the plane which minimizes the sum of squares of the distance to $(0,1), (0,0)$ and $(2,0)$. Explain why your answer is a minimum & not a maximum.

→ Integrals

35–40 Find the volume of the given solid.

35. Under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$

36. Under the surface $z = x^2 y$ and above the triangle in the xy-plane with vertices $(1, 0)$, $(2, 1)$, and $(4, 0)$

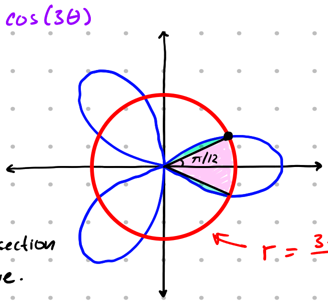
37. The solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 2, 0)$, and $(2, 2, 0)$

38. Bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$

39. One of the wedges cut from the cylinder $x^2 + 9y^2 = a^2$ by the planes $z = 0$ and $z = mx$

40. Above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$

$r(\theta) = 3\cos(3\theta)$



Write a double integral representing the area in the picture.

$r(\theta) = 3\cos(3\theta) = 0 \Rightarrow \theta = \pm \frac{\pi}{6}$

1. Find the intersection of circle + curve.

$r = \frac{3\sqrt{2}}{2}$ $\quad x^2 + y^2 = \frac{9 \cdot 2}{4} = 9/2$

$\cancel{3}\cos(3\theta) = \frac{\cancel{3}\sqrt{2}}{2}$ $\quad \cos(3\theta) = \frac{\sqrt{2}}{2}$ $\quad 3\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{12}$

$\theta = -\frac{\pi}{12}$

$$\boxed{2\int_{\pi/12}^{\pi/6}\int_0^{3\cos(3\theta)} r\,dr\,d\theta + \int_{-\pi/12}^{\pi/12}\int_0^{3\sqrt{2}/2} r\,dr\,d\theta}$$

1. Find the max/min of $f(x,y) = 18x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$ $\quad (\pm\frac{3}{2}, 0) \quad (0, \pm 3)$

$g(x,y) = 4x^2 + y^2$

$\nabla f = \langle 36x, 2y \rangle \qquad \nabla f = \lambda \nabla g$

$\nabla g = \langle 8x, 2y \rangle$

$36x = 8\lambda x$

$2y = 2\lambda y$

$4x^2 + y^2 = 9$

$\lambda = 1$ & $y \neq 0$ $\quad$ or $\quad$ $\lambda = 4$ & $x \neq 0$

Case 1: $y \neq 0,\ \lambda = 1$

$36x = 8\lambda x \Rightarrow 36x = 8x \quad 36x - 8x = 0 \Rightarrow x = 0$

$4x^2 + y^2 = 9 \quad y^2 = 9 \Rightarrow y = \pm 3 \Rightarrow \boxed{(0, \pm 3)}$

Case 2:

$2y = 8y \Rightarrow y = 0$

$4x^2 + y^2 = 9 \Rightarrow 4x^2 = 9$

$\Rightarrow x^2 = 9/4 \Rightarrow x = \pm 3/2$

$\Rightarrow \boxed{(\pm \frac{3}{2}, 0)}$

min

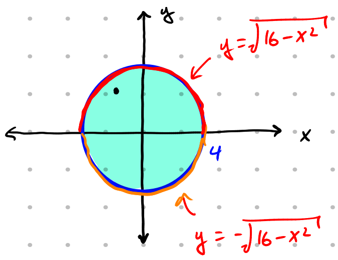
$f(0, \pm 3) = 9$

$f(\pm 3/2, 0) = \cancel{18}^{9} \cdot \frac{9}{\cancel{4}_{2}} = \frac{81}{2} > 9$

max

Absolute Max/Min

1. Find the absolute max/min of $f(x,y) = x^2 + y^2 + 4x - 6y$ on the domain $x^2 + y^2 \le 16$



$f_x = 2x + 4 = 0 \Rightarrow x = -2$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

$f(-2, 3) = -4 - 9$

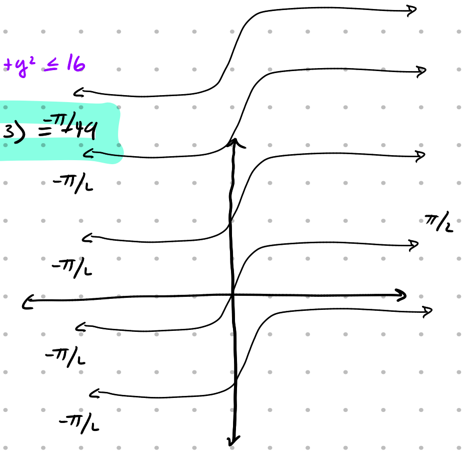
$\vec{r}(\theta) = \langle 4\cos\theta, 4\sin\theta \rangle \quad 0 \le \theta \le 2\pi$

$f(\theta) = 16 + 16\cos\theta - 24\sin\theta$

$f'(\theta) = -16\sin\theta - 24\cos\theta = 0$

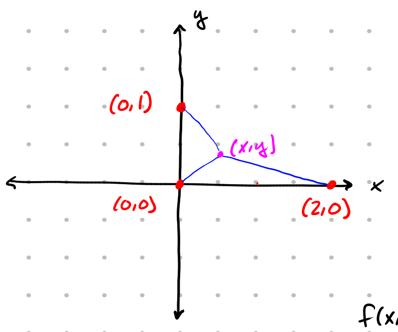
$2\sin\theta + 3\cos\theta = 0 \leadsto \tan^{-1}(-3/2) = \theta$

$-.9$ &



1. Find the point (x,y) of the plane which minimizes the sum of squares of the distance to $(0,1), (0,0)$ and $(2,0)$

Explain why your answer is a minimum + not a maximum.



$d_1 = \text{distance}^2$ from (x,y) to $(0,0)$ is $x^2 + y^2$

$d_2 = \text{distance}^2$ from (x,y) to $(0,1)$ is $x^2 + (y-1)^2$

$d_3 = \text{distance}^2$ from (x,y) to $(2,0)$ is $(x-2)^2 + y^2$

Minimize $d_1 + d_2 + d_3$.

$f(x,y) = x^2 + y^2 + x^2 + (y-1)^2 + (x-2)^2 + y^2$

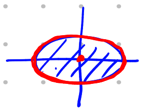
$= x^2 + y^2 + x^2 + y^2 - 2y + 1 + x^2 - 4x + 4 + y^2$

$= 3x^2 + 3y^2 - 4x - 2y + 5$

$(\frac{2}{3}, \frac{1}{3})$.

Problem 5

Find extreme values of $f(x,y) = e^{-xy}$ on $x^2+4y^2 \leq 1$ $\quad x^2+4y^2 = 1$



$f_x(x,y) = \frac{d}{dx}\left(e^{-xy}\right) = -y\,e^{-xy} = 0$

$f_y(x,y) = \frac{d}{dy}\left(e^{-xy}\right) = -x\,e^{-xy} = 0$

$\Rightarrow$ $x=0$ & $y=0$ So $(0,0)$ is the only critical point of $f(x,y)$

$\boxed{f(0,0) = 1}$

$\vec{r}(\theta) = \langle x = \cos\theta,\ y = \frac{1}{2}\sin(\theta)\rangle$, $0 \leq \theta \leq 2\pi$

$f(\theta) = e^{-\frac{1}{2}\sin\theta\cos\theta} = e^{-\frac{1}{4}\sin(2\theta)}$

$\frac{d}{d\theta} f(\theta) = e^{-\frac{1}{4}\sin(2\theta)} \cdot \frac{d}{d\theta}\left(-\frac{1}{4}\sin(2\theta)\right) = -\frac{1}{2} e^{-\frac{1}{4}\sin(2\theta)}(\cos(2\theta))$

"$\Leftrightarrow$" = if & only if

$-\frac{1}{2} e^{-\frac{1}{4}\sin(2\theta)}(\cos(2\theta)) = 0$ when $\cos(2\theta) = 0 \Leftrightarrow \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

$\Rightarrow$ The critical points on the boundary are at

$f(\pi/4)$ & $f(3\pi/4)$

< 1 $\quad \frac{1}{e^{1/4}}$

$e^{-\frac{1}{4}\sin(2\theta)}$ $\quad f(\pi/4) = e^{-\frac{1}{4}\sin(\pi/2)} = e^{-1/4} \rightarrow$ min $\quad .78$

$f(3\pi/4) = e^{-\frac{1}{4}\sin(3\pi/2)} = e^{1/4} \rightarrow$ max $\quad .12$

$\cancel{\boxed{f(0,0) = 1}}$

Problem 1

Evaluate the integral. You may find it helpful to make a change of coordinates.

$$\int\int_R (x+y)e^{x^2-y^2}\,dA$$

where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$

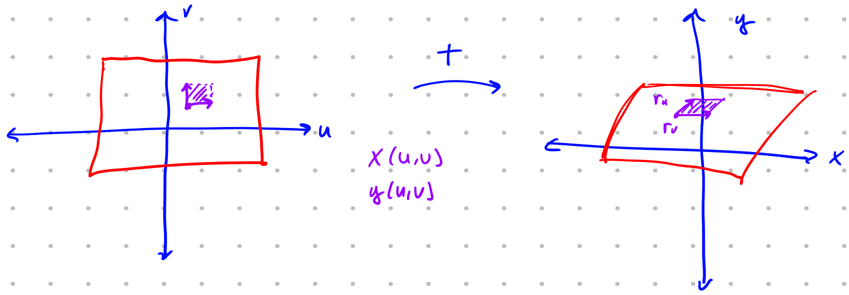
$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

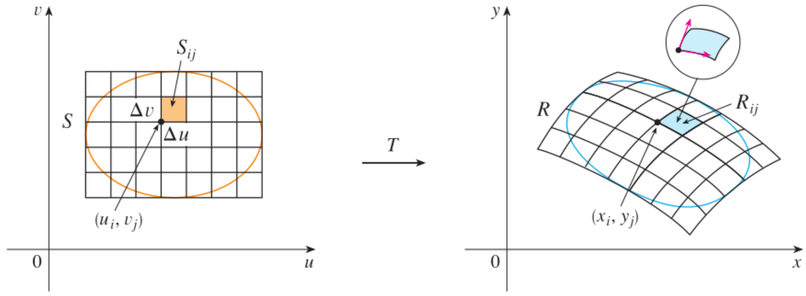
$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

absolute value

determinant of Jacobian matrix



Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.)



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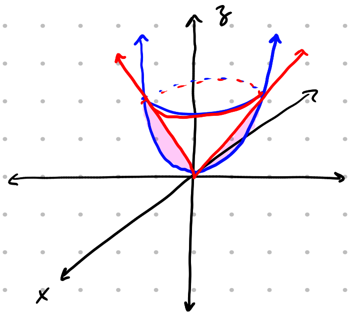

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta = 2\pi\int_0^1 r(r-r^2)\,dr = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{2\pi}{3}-\frac{2\pi}{4}$$