Tuesday, August 2, 2022

MATH 164 Lecture Notes

Review #2

1. Find a function f such that $\nabla f = \vec{F}$

FTLI: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Problem 1

Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = \left\langle y\sin(z)e^z, \underline{x\sin(z)e^z}, xye^z\left(\cos(z)+\sin(z)\right)\right\rangle$$

and C is given by

$$\mathbf{r}(t) = \left\langle (2+\sin(t))\cos(t), (2+\sin(t))\sin(t), \frac{t}{\pi}+\cos(t)\right\rangle \qquad \underset{a}{0} \le t \le \underset{b}{5\pi}$$

Hint: Apply the Fundamental Theorem of Line Integrals.

$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$

$f_x(x,y,z) = y\sin(z)e^z \rightsquigarrow xy\sin(z)e^z + K_1(y,z)$

$f_y(x,y,z) = x\sin(z)e^z \rightsquigarrow xy\sin(z)e^z + K_2(x,z)$ $\quad \Rightarrow K_1(y,z) = K_2(x,z) = K_3(x,y) = K$

$f_z(x,y,z) = xye^z(\cos(z)+\sin(z)) \rightsquigarrow xy\sin(z)e^z + K_3(x,y)$

$f = xy\sin(z)e^z + K$

$\vec{r}(5\pi) = \langle 0, 3, 5\rangle \qquad f(\vec{r}(5\pi)) = f(\overset{x}{0},\overset{y}{3},\overset{z}{5}) = 0 \qquad \Rightarrow \int_C \vec{F}\cdot d\vec{r} = 0$

$\vec{r}(0) = \langle 2, 0, 1\rangle \qquad f(\vec{r}(0)) = f(2,0,1) = 0$

FTLI: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

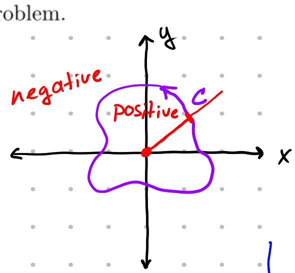
Problem 2

Find the positively oriented, simple, closed curve C (in the xy−plane) that maximizes the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \left\langle \overset{P}{y^5 + 2y^2}, \overset{Q}{-x^3 + 4yx + 9x}\right\rangle.$$

$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P\,dx + Q\,dy$

Hint: Recall that $\mathbf{F}\cdot d\mathbf{r} = P\,dx + Q\,dy$ for vector fields of the form $\mathbf{F} = \langle P, Q\rangle$. This is a <u>Green's theorem</u> problem.

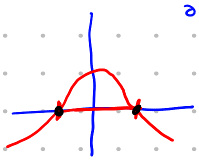


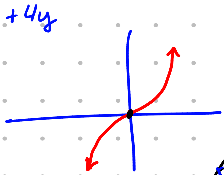
$f(x,y)$ $\qquad \iint_D (\text{positive stuff})\, dA$

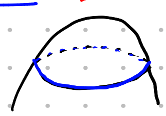
$\frac{\partial Q}{\partial x} = -3x^2 + 4y + 9 \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -3x^2 + 4y + 9 - 5y^4 - 4y$

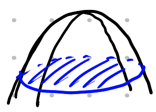
$\frac{\partial P}{\partial y} = 5y^4 + 4y \qquad g(x,y) = -3x^2 - 5y^4 + 9 = 0 \qquad z = y^4$

$$\boxed{C = \{(x,y) \in \mathbb{R}^2 : 3x^2 + 5y^4 = 9\}}$$





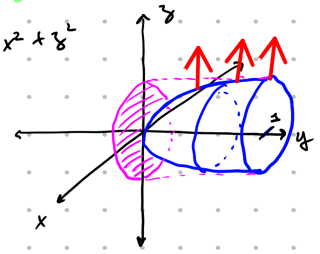




Problem 3

Evaluate the surface integral: $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z)=\langle 0,y,-z\rangle$ and S is the part of the paraboloid $y=x^2+z^2$ with $0\le y\le 1$. **Hint:** There are several ways to do this problem. Recall that $\int_0^{2\pi}\sin^2 x\,dx = \int_0^{2\pi}\cos^2 x\,dx=\pi$. This might be useful depending on which method you choose.

$1 = x^2+z^2$



$$\iint_S \vec F\cdot d\vec S = \iint_D \vec F(\vec r(u,v))\cdot\underbrace{(\vec r_u\times\vec r_v)}_{\nabla f}dA$$

$\boxed{f(x,y,z) = x^2+z^2-y = 0}$

$\vec r(x,z) = \langle x, x^2+z^2, z\rangle$ $\quad \boxed{\nabla f = \langle 2x,-1,2z\rangle} = \boxed{\vec r_x\times\vec r_z}$

$\vec F(\vec r(x,z)) = \langle 0, x^2+z^2, -z\rangle$

$S \leftrightarrow f(x,y,z)=k$

$\boxed{\text{* If your suface } S \text{ is the level surface of } f(x,y,z), \text{ then } \vec r_u\times\vec r_v = \nabla f.}$

$$\vec F(\vec r(x,z))\cdot(\vec r_x\times\vec r_z) = \langle 0, x^2+z^2, -z\rangle\cdot\langle 2x,-1,2z\rangle$$
$$= -x^2-z^2-2z^2 = -x^2-3z^2$$

$$\iint (-x^2-3z^2)$$
$$\downarrow$$
$$\int_0^{2\pi}\int_0^1 r(-r^2\cos^2\theta - 3r^2\sin^2\theta)\,dr\,d\theta$$

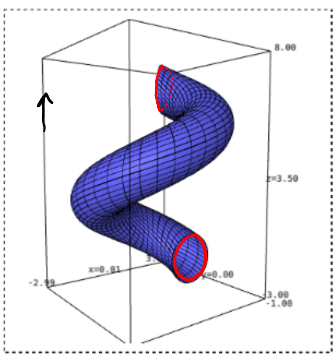
D is the domain where the parameters live.

1. Parameterize S to get $\vec r(u,v)$
2. Calculate $\vec r_u\times\vec r_v$
3. Calculate $\vec F(\vec r(u,v))$
4. Calculate $\vec F(\vec r(u,v))\cdot(\vec r_u\times\vec r_v)$
5. Integrate $\iint_D \vec F(\vec r(u,v))\cdot(\vec r_u\times\vec r_v)\,dA$

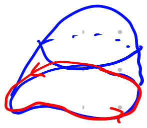
* D = Domain where u + v live.

$$= \int_0^{2\pi}\int_0^1(-r^3\cos^2\theta - 3r^3\sin^2\theta)\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(-\frac14\cos^2\theta - \frac34\sin^2\theta\right)d\theta$$
$$= -\frac14\pi - \frac34\pi = -\pi$$

Part b (7 points): Recall the noodle shape whose parametric equations are $\mathbf{r}(u,v)=\langle(2+\sin v)\cos u,(2+\sin v)\sin u, u+\cos v\rangle$. Calculate $\iint_S\mathbf F\cdot d\mathbf S$ where S is the portion of the noodle with $0\le u\le 2\pi$, $0\le v\le 2\pi$ and $\mathbf F(x,y,z)=\langle 0,0,1\rangle$. **Hint:** You might want to apply Stokes' theorem.



$$\iint_S(\vec\nabla\times\vec G)\cdot d\vec S = \int_{\partial S}\vec G\cdot d\vec r$$



$\vec F(u,v) = \langle(2+\sin v)\cos u, (2+\sin v)\sin u, u+\cos(v)\rangle$

$u=0$ & $u=2\pi$

$\boxed{\text{* Hint: } \vec G = \langle 0,x,0\rangle \text{ satisfies } \vec\nabla\times\vec G = \vec F}$

$\iint_S(\vec\nabla\times\vec G)\cdot d\vec S = \int_{\partial S}\vec G\cdot d\vec r$

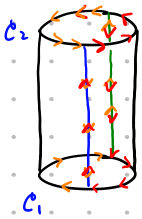
$\vec r_1(v) = \vec r(2\pi,v) = \langle 2+\sin v, 0, 2\pi+\cos v\rangle$

$\vec r_1'(v) = \langle\cos v, 0, -\sin v\rangle$ $\qquad \vec G(\vec r(v))$

$\vec r_2(v) = \vec r(0,v) = \langle 2+\sin v, 0, \cos v\rangle$

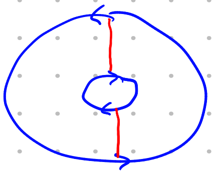
$\vec r_2'(v) = \langle\cos v, 0, -\sin v\rangle$

$$\iint_S\vec F\cdot d\vec S = \iint_S(\vec\nabla\times\vec G)\cdot d\vec S = \int_{\partial S}\vec G\cdot d\vec r = \int_0^{2\pi}\langle 0,2+\sin v,0\rangle\cdot\langle\cos v,0,-\sin v\rangle\,dv = 0$$
$$+\int_0^{2\pi}\langle 0,2+\sin v,0\rangle\cdot\langle\cos v,0,-\sin v\rangle\,dv = 0$$
$$= 0+0$$



$$\iint_S (\vec{\nabla}\times\vec{G})\cdot d\vec{S} = \int_{\partial S} \vec{G}\cdot d\vec{r} = \int_{C_1} \vec{G}\cdot d\vec{r} - \int_{C_2} \vec{G}\cdot d\vec{r}$$

*When you have two boundary components.



Problem 5

Calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = \langle x^2yz, xy^2z, xyz^2\rangle$ and S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$ where a, b, and c are positive numbers.
Hint: Use the Divergence Theorem.

$\frac{3}{4}a^2b^2c^2$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \vec{\nabla}\cdot\vec{F}\,dV$$

$$\vec{\nabla}\cdot\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle\cdot\langle x^2yz, xy^2z, xyz^2\rangle = 2xyz\cdot 3 = 6xyz$$

$$6\int_0^c\int_0^b\int_0^a xyz\,dx\,dy\,dz = 6\cdot\frac{x^2}{2}\Big|_0^a\cdot\frac{y^2}{2}\Big|_0^b\cdot\frac{z^2}{2}\Big|_0^c = \frac{6}{8}a^2b^2c^2 = \frac{3}{4}a^2b^2c^2$$

* Lagrange multipliers
* min/max
* Double + triple integrals (setup)