MATH 164 Practice Problems

Monday, August 1, 2022

Find the intersection between the surfaces

1. $\sqrt{x^2+y^2} = z \quad \& \quad z = 1+y$

Solution: $x^2+y^2 = y^2+2y+1$

$x^2 = 2y+1 \Rightarrow y = \frac{1}{2}x^2 - \frac{1}{2}$

$\vec{r}(t) = \langle t, \frac{1}{2}t^2 - \frac{1}{2}, \frac{1}{2}t^2 + \frac{1}{2} \rangle$

2. $x^2+y^2+z^2 = 1 \quad \& \quad y = 2z+1$

Solution: $y^2 = (2z+1)(2z+1) = 4z^2+4z+1$

$\Rightarrow x^2+z^2+4z^2+4z+1 = 1$

$\Rightarrow x^2+5z^2+4z = 1$

$x^2 + 5(z+\frac{2}{5})^2 = \frac{9}{5}$ (complete the square)

Let $x = a\cos(t)$ $\quad z = a\frac{1}{\sqrt{5}}\sin(t) - \frac{2}{5}$

then $a^2\cos^2(t) + 5\left(\frac{a^2}{5}\sin^2(t)\right) = \frac{9}{5}$

$a^2\cos^2(t) + a^2\sin^2(t) = \frac{9}{5} \Rightarrow a^2 = \frac{9}{5} \Rightarrow a = \frac{3}{\sqrt{5}}$

Finally, we parameterize:

$\vec{r}(t) = \langle \frac{3}{\sqrt{5}}\cos(t), \frac{6}{5}\sin(t) - \frac{4}{5} + 1, \frac{3}{5}\sin(t) - \frac{2}{5} \rangle$

$\vec{r}(t) = \langle \frac{3}{\sqrt{5}}\cos(t), \frac{6}{5}\sin(t) + \frac{1}{5}, \frac{3}{5}\sin(t) - \frac{2}{5} \rangle$

3. $4x^2+4y^2+z^2 = 16 \quad x = z^2$

Solution:

We let $z = t$. Then we have:

$4t^4 + 4y^2 + t^2 = 16$

$\Rightarrow 4y^2 = 16 - t^2 - 4t^4$

$y^2 = \frac{16-t^2-4t^4}{4}$

$\Rightarrow y = \pm\frac{1}{2}\sqrt{16-t^2-4t^4}$

$\vec{r}(t) = \langle t^2, \pm\frac{1}{2}\sqrt{16-t^2-4t^4}, t \rangle$

Limit

1. $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

Solution: when $y=0$, $\lim_{x\to 0} \frac{x^2}{x^2} = 1$

when $x=y$, $\lim_{x\to 0} \frac{4x^2}{2x^2} = 2$

$\Rightarrow$ DNE

2. $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Solution: Let $x^2+y^2 = r$. We have: $\lim_{r\to 0^+} \frac{\sin(r)}{r} = 1$

3. $\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$

Solution: Note that for <u>all lines</u> through the origin, $y = mx$, we have

$$\lim_{x\to 0} \frac{mx^2}{x(m+1)} = \frac{mx}{m+1} = 0$$

However, if we let $y = x^2 - x$, we get

$$\lim_{x\to 0} \frac{x(x^2-x)}{x+x^2-x} = \frac{x^3-x^2}{x^2} = x-1 = -1 \neq 0$$

$\Rightarrow$ The limit does not exist.

<u>Multiple Integrals</u>

1. Write an integral representing the volume below $z = 6-x$, above $z = -\sqrt{4x^2+4y^2}$ + inside $x^2+y^2 = 3$.

Solution:

$$\int_0^{2\pi}\int_0^{3}\int_{-2r}^{6-r\cos\theta} r\,dr\,d\theta$$

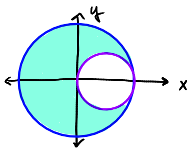
2. Write an integral representing the volume below $z = 6-x$, above $z = -\sqrt{4x^2+4y^2}$ + inside $x^2+y^2 = 3$, $x \le 0$.

Solution:

$$\int_{-\pi/2}^{\pi/2}\int_0^{3}\int_{-2r}^{6-r\cos\theta} r\,dr\,d\theta$$

3. Write an integral representing the volume below $-x^2-y^2+4 = z$, above $z=0$, + outside $(x-1)^2+y^2 = 1$

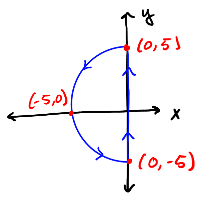
Solution:



$$\int_0^{2\pi}\int_0^{2}\int_0^{-r^2+4} r\,dz\,dr\,d\theta - \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{-r^2+4} r\,dz\,dr\,d\theta$$

Green's theorem: $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

Use Green's theorem to calculate $\oint_C yx^2\,dx - x^2\,dy$ where C is given by the picture:



Solution:

Use Green's theorem:

$$\int_{\pi/2}^{3\pi/2}\int_0^{5} (-2r\cos\theta - r^2\cos^2\theta)\, r\,dr\,d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \left(-2\cdot\tfrac{1}{3}5^3\cos\theta + \tfrac{1}{4}5^4\cos^2\theta\right)d\theta$$

$$= \frac{-2\cdot 5^3}{3}\sin\theta\Big|_{\pi/2}^{3\pi/2} + \frac{5^4}{4}\cdot\frac{\pi}{2}$$

$$= \frac{-4\cdot 5^3}{3} + \frac{5^4\pi}{8} \quad ✓$$

Evaluate $\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$ where $\vec{F} = \langle 2yz, -x+3y, x^2+z\rangle$ & S is the cylinder $x^2+y^2=1,\ 0\le z\le 1$

Solution: We might be able to use Stokes' theorem here, but let's do the surface integral directly.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & -x+3y & x^2+z \end{vmatrix} = \langle 0, 2y-2x, -1-2z\rangle$$

parameterize S: $\vec{r}(t,z) = \langle \cos(t), \sin(t), z\rangle \qquad 0\le z\le 1,\ 0\le t\le 2\pi$

$\vec{r}_t = \langle -\sin t, \cos t, 0\rangle$

$\vec{r}_z = \langle 0, 0, 1\rangle$

$$\vec{r}_t\times\vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos(t), \sin(t), 0\rangle$$

$$\vec{F}(\vec{r}(t))\cdot(\vec{r}_t\times\vec{r}) = \langle 0, 2\sin(t)-2\cos(t), -1-2z\rangle\cdot\langle\cos(t),\sin(t),0\rangle$$

$$= 2\sin^2(t) - 2\sin(t)\cos(t)$$

Then integrate

$$\int_0^{2\pi}\int_0^1 (-2\sin(t)\cos(t))\,dz\,dt + \int_0^{2\pi}\int_0^1 2\sin^2(t)\,dz\,dt$$

$$= \int_0^{2\pi}(-2\sin(t)\cos(t))\,dt + \int_0^{2\pi} 2\sin^2(t)\,dt$$

$$= \int_0^{2\pi}(-\sin(2t))\,dt + \int_0^{2\pi}(1-\cos(2\theta))\,dt$$

$$= \frac{1}{2}\cos(2t)\Big|_0^{2\pi} + t\Big|_0^{2\pi} - \frac{1}{2}\sin(2t)\Big|_0^{2\pi}$$

$$= \frac{1}{2}(\cos(4\pi)-\cos(0)) + 2\pi - \frac{1}{2}(\sin(4\pi)-\sin(0)) = 2\pi$$

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