Thursday July 7, 2022

MATH 164 Lecture Notes

* Midterm 1 on Monday

* Probably have a Tuesday morning midterm.

Review!

Sections

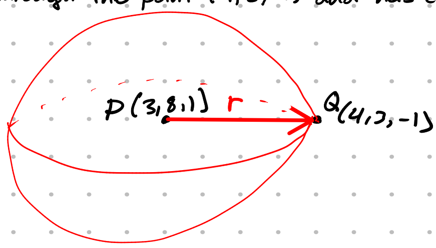
12.1 3d Coordinates
12.2 vectors
12.3 dot products
12.4 cross products
12.5 lines + planes
12.6 cylinders + quadric surfaces
13.1 vector functions + space curves
13.2 derivatives + integrals of vector functions
13.3 arc length + curvature
14.1 functions of several variables
14.2 limits + continuity.

12.1 3d Coordinates

Problem 15: Find an equation of the sphere that passes through the point $(4,3,-1)$ and has center $(3,8,1)$.

Sphere: Sphere w/ center (a,b,c) + radius r:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$



Need to find $r = \text{distance}(P,Q)$

$$r = \sqrt{(4-3)^2 + (3-8)^2 + (-1-1)^2}$$

12.2 vectors

Problem 26: Find a vector that has the same direction as $\langle 6,2,-3\rangle$ but has length 4.

Problem 29: If $\vec{v}$ lies in the first quadrant and makes an angle $\pi/3$ with the positive x-axis and $|\vec{v}|=4$, find $\vec{v}$ in component form.

Problem 48: If $\vec{r} = \langle x,y\rangle$, $\vec{r}_1 = \langle x_1,y_1\rangle$, $\vec{r}_2 = \langle x_2,y_2\rangle$, describe the set of all points (x,y) such that $|\vec{r}-\vec{r}_1| + |\vec{r}-\vec{r}_2| = k$ where $k > |\vec{r}_1 - \vec{r}_2|$.

12.3 dot products

Problem 27: Find a vector that is orthogonal to both $\vec{i}+\vec{j}$ and $\vec{i}+\vec{k}$

Problem 45: Show that the vector $\vec{v} = \vec{b} - \text{proj}_{\vec{a}}\vec{b}$ is orthogonal to $\vec{a}$.

12.4 cross products

Problem 45: Let P be a point not on the line L that passes through the points Q + R. Show that the distance d from P to L is

$$d = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$$

where $\vec{a} = \overrightarrow{QR}$ $\vec{b} = \overrightarrow{QP}$

12.5 lines + planes

6–12 Find parametric equations and symmetric equations for the line.

6. The line through the origin and the point $(4, 3, -1)$
7. The line through the points $\left(0, \frac{1}{2}, 1\right)$ and $(2, 1, -3)$
8. The line through the points $(1, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$
9. The line through the points $(-8, 1, 4)$ and $(3, -2, 4)$
10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$
11. The line through $(-6, 2, 3)$ and parallel to the line $\frac{1}{2}x = \frac{1}{3}y = z + 1$
12. The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

23–40 Find an equation of the plane.

23. The plane through the origin and perpendicular to the vector $\langle 1, -2, 5\rangle$

24. The plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

25. The plane through the point $\left(-1, \frac{1}{2}, 3\right)$ and with normal vector $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

26. The plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$

27. The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$

28. The plane through the point $(3, -2, 8)$ and parallel to the plane $z = x + y$

29. The plane through the point $\left(1, \frac{1}{2}, \frac{1}{3}\right)$ and parallel to the plane $x + y + z = 0$

30. The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

31. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$

Point: $(3, -2, 8)$ $\quad 0 = -z + x + y$

parallel to $z = x + y$ $\quad \vec{n} = \langle 1, 1, -1\rangle$

normal

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

position vector of a point in the plane

"variable vector $\langle x, y, z\rangle$"

$\langle 1, 1, -1\rangle \cdot \langle x-3, y+2, z-8\rangle$

$= \boxed{x - 3 + y + 2 - z + 8 = 0}$

Problem 2

a) Let $l_1(t) = \langle t+1, 3t, 2t-1\rangle$ and $l_2(s) = \langle s, s+1, s+2\rangle$ be two lines. Show that l_1 and l_2 are skew.

not parallel & non-intersecting

b) What is the distance between l_1 and l_2?

a) $\vec{v}_1 = \langle 1, 3, 2\rangle$
$\vec{v}_2 = \langle 1, 1, 1\rangle$ $\rightsquigarrow$ clearly not parallel

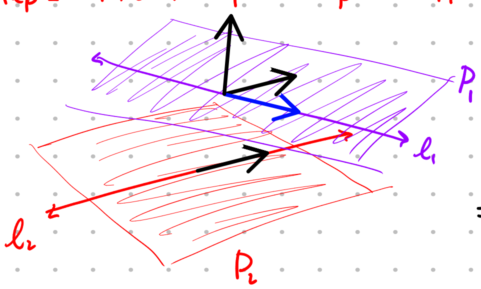
$t + 1 = s$
$3t = s + 1$ $\rightsquigarrow$ no solutions.
$2t - 1 = s + 2$

b) Hint: Distance between a point (x_1, y_1, z_1) & a plane $ax + by + cz + d = 0$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$1 \cdot 1 - 0 + 2 = 3$

Step 1: Find two parallel planes P_1 & P_2 such that $l_1(t) \in P_1\ \forall t$ and $l_2(t) \in P_2$



$P_1 \parallel P_2 \quad \vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 1, 1, -2\rangle$

$l_1(0) = \langle 1, 0, -1\rangle$ (x_1, y_1, z_1)

$\Rightarrow P_2: \boxed{x - y - 2z + 5 = 0}$

$l_2(0) = \langle 0, 1, 2\rangle$

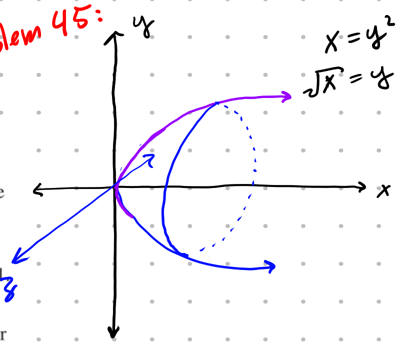
$$D = \frac{|1 + 2|}{\sqrt{1+1+4}} \quad = 6$$

$\boxed{= \frac{3}{\sqrt{6}}}$

Step 2: Calculate the distance between P_1 & P_2.

12.6 cylinders & quadric surfaces

Problem 45:



45. Find an equation for the surface obtained by rotating the curve $y = \sqrt{x}$ about the x-axis.

46. Find an equation for the surface obtained by rotating the line $z = 2y$ about the z-axis.

47. Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.

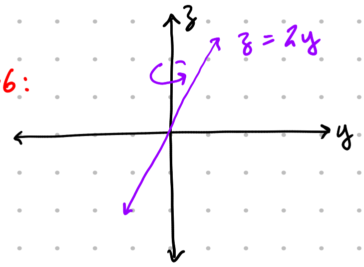
48. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$\frac{x}{c} = \frac{z^2}{a^2} + \frac{y^2}{b^2}$

$\boxed{x = z^2 + y^2}$

Problem 46:



Cone $\boxed{\frac{z^2}{4} = x^2 + y^2}$

$\frac{z}{c} = \frac{y}{b} \qquad z = 2y$

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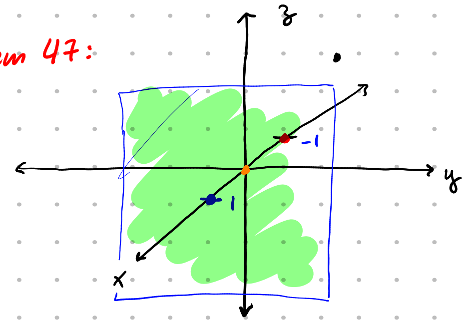
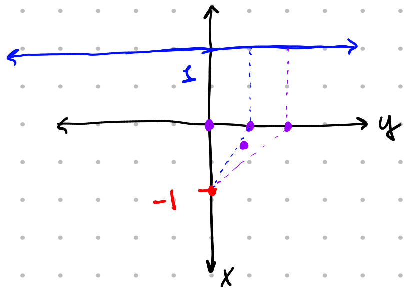
Problem 47:





• The distance from (x,y,z) to the plane is: $|x-1|$

• The distance from (x,y,z) to $(-1,0,0)$:

$$\sqrt{(x+1)^2 + y^2 + z^2}$$

• $S = \{(x,y,z) : (x-1)^2 = (x+1)^2 + y^2 + z^2 \}$

$$\cancel{x^2} - 2x \cancel{+1} = \cancel{x^2} + 2x \cancel{+1} + y^2 + z^2$$

$$0 = 4x + y^2 + z^2$$

$$-4x = y^2 + z^2$$

$$-1000 = x$$

$$4000 = y^2 + z^2$$

$\boxed{\text{parabaloid}}$

13.1: vector functions & space curves

42–46 Find a vector function that represents the curve of intersection of the two surfaces.

42. The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$

43. The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$

44. The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$

hyperbolic paraboloid.

45. The ~~hyperboloid~~ $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$

46. The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geqslant 0$, and the cylinder $x^2 + z^2 = 1$

49. If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) The curves might intersect, but we need to know whether the objects are in the same position *at the same time.* Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geqslant 0$. Do the particles collide?

46//

$x^2 + y^2 + 4z^2 = 4 \qquad y \geq 0$

$x^2 + z^2 = 1 \qquad \boxed{x = \cos(t) \quad z = \sin(t)}$

$y = \sqrt{4 - x^2 - 4z^2}$

$\Downarrow$

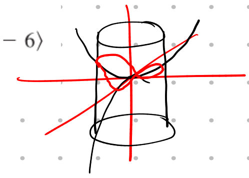
$y = \sqrt{4 - \cos^2(t) - 4\sin^2(t)}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}(t) = \langle \cos(t), \sqrt{4 - \cos^2(t) - 4\sin^2(t)}, \sin(t) \rangle$

46// hyperboloid: $z = x^2 - y^2$ & $x^2 + y^2 = 1$

$x = \cos(t) \quad y = \sin(t)$



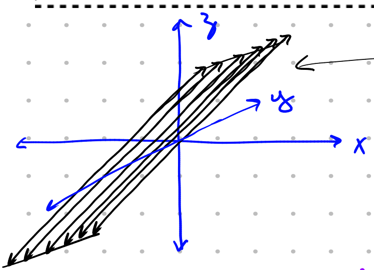
$\vec{r}(t) = \langle \cos(t), \sin(t), \cos^2(t) - \sin^2(t) \rangle$

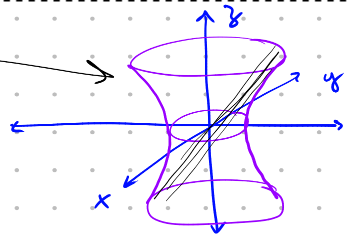
hyperboloid of 1 sheet

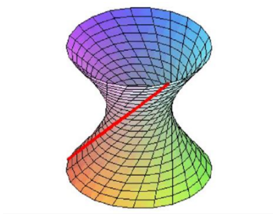
Problem 1

Find a parametric equation for the curve (or curves) of intersection between the surface $\frac{x^2}{9} + y^2 - \frac{z^2}{9} = 1$ and the plane $\boxed{x = z}$

$x = t \qquad z = t$







$y^2 = 1 \Rightarrow y = \pm 1$

$\vec{r}_1(t) = \langle t, 1, t \rangle$ "describe the curves"

$\vec{r}_2(t) = \langle t, -1, t \rangle$

13.2: derivatives + integrals of vector functions

13.3 arc length + curvature

14.1 functions of several variables

23–26 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

23. $x = t^2 + 1, \quad y = 4\sqrt{t}, \quad z = e^{t^2 - t}; \quad (2, 4, 1)$

24. $x = \ln(t + 1), \quad y = t \cos 2t, \quad z = 2^t; \quad (0, 0, 1)$

25. $x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad z = e^{-t}; \quad (1, 0, 1)$

26. $x = \sqrt{t^2 + 3}, \quad y = \ln(t^2 + 3), \quad z = t; \quad (2, \ln 4, 1)$

13.3: arc length + curvature

1–6 Find the length of the curve.

1. $\mathbf{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle, \quad -5 \leqslant t \leqslant 5$

2. $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle, \quad 0 \leqslant t \leqslant 1$

3. $\mathbf{r}(t) = \sqrt{2}\, t\, \mathbf{i} + e^t\, \mathbf{j} + e^{-t}\, \mathbf{k}, \quad 0 \leqslant t \leqslant 1$

4. $\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + \ln \cos t\, \mathbf{k}, \quad 0 \leqslant t \leqslant \pi/4$

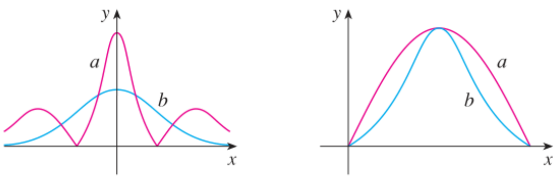
5. $\mathbf{r}(t) = \mathbf{i} + t^2\, \mathbf{j} + t^3\, \mathbf{k}, \quad 0 \leqslant t \leqslant 1$

6. $\mathbf{r}(t) = t^2\, \mathbf{i} + 9t\, \mathbf{j} + 4t^{3/2}\, \mathbf{k}, \quad 1 \leqslant t \leqslant 4$

32. Find an equation of a parabola that has curvature 4 at the origin.

38–39 Two graphs, a and b, are shown. One is a curve $y = f(x)$ and the other is the graph of its curvature function $y = \kappa(x)$. Identify each curve and explain your choices.

38. **39.**



14.1: functions of several variables

45–52 Draw a contour map of the function showing several level curves.

45. $f(x, y) = x^2 - y^2$

46. $f(x, y) = xy$

47. $f(x, y) = \sqrt{x} + y$

48. $f(x, y) = \ln(x^2 + 4y^2)$

49. $f(x, y) = ye^x$

50. $f(x, y) = y - \arctan x$

51. $f(x, y) = \sqrt[3]{x^2 + y^2}$

52. $f(x, y) = y/(x^2 + y^2)$

14.2: limits & continuity

7. $\lim_{(x, y)\to(\pi, \pi/2)} y \sin(x - y)$

8. $\lim_{(x, y)\to(3, 2)} e^{\sqrt{2x-y}}$

9. $\lim_{(x, y)\to(0, 0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$

10. $\lim_{(x, y)\to(0, 0)} \frac{5y^4\cos^2 x}{x^4 + y^4}$ DNE

11. $\lim_{(x, y)\to(0, 0)} \frac{y^2\sin^2 x}{x^4 + y^4}$

12. $\lim_{(x, y)\to(1, 0)} \frac{xy - y}{(x - 1)^2 + y^2}$ Exists

13. $\lim_{(x, y)\to(0, 0)} \frac{xy}{\sqrt{x^2 + y^2}}$

14. $\lim_{(x, y)\to(0, 0)} \frac{x^3 - y^3}{x^2 + xy + y^2}$ Exists

15. $\lim_{(x, y)\to(0, 0)} \frac{xy^2\cos y}{x^2 + y^4}$

16. $\lim_{(x, y)\to(0, 0)} \frac{xy^4}{x^4 + y^4}$

17. $\lim_{(x, y)\to(0, 0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

18. $\lim_{(x, y)\to(0, 0)} \frac{xy^4}{x^2 + y^8}$

10) $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$

$x=0 \quad \lim_{y\to 0} \frac{5\cancel{y^4}}{\cancel{y^4}} = 5$

$y=0 \quad \lim_{x\to 0} \frac{0}{x^4}$ DNE

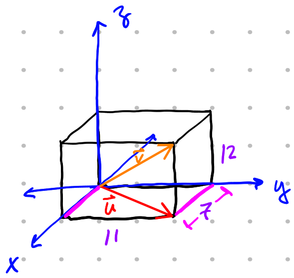
$x=y \quad \lim_{y\to 0} \frac{5\cancel{y^4}\cos^2(y)}{2\cancel{y^4}} = \frac{5}{2} \neq 5$ DNE

12) $\lim_{(x,y)\to(1,0)} \frac{xy-y}{(xy-1)^2+y^2} = \lim_{(x,y)\to(1,0)} \frac{y(x-1)}{x^2y^2 - 2xy + 1 + y^2}$ "squeez theorem"

$\left| \frac{y(x-1)}{x^2y^2 - 2xy + 1 + y^2} \right| \leq \left| \frac{\cancel{y}(x-1)}{-2x\cancel{y}} \right| = \left| \frac{(x-1)}{-2x} \right| \quad \lim_{x\to 1} \left| \frac{x-1}{-2x} \right| = 0$ ✓

14) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{(x+y)^2} = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2+2xy}$ $x = r\cos\theta \quad y = r\sin\theta$

$= \lim_{r\to 0} \frac{r^3\cos^3\theta - r^3\sin^3\theta}{\cancel{r^2} + 2r^2\cos\theta\sin\theta} = \lim_{r\to 0} \frac{r(\cos^3\theta - \sin^3\theta)}{1 + 2\cos\theta\sin\theta} = 0$

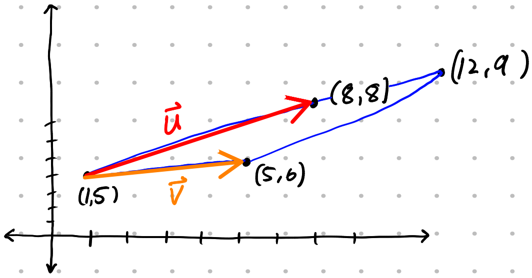


Angle between $\vec{u}$ + $\vec{v}$

$$\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \cos(\theta)$$

$\vec{u} = \langle 7, 11, 0 \rangle$

$\vec{v} = \langle 7, 11, 12 \rangle$

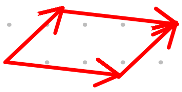


$\vec{u} = \overrightarrow{(1,5).\,(8,8)} = \langle 8-1, 8-5 \rangle = \langle 7, 3 \rangle$

$\vec{v} = \overrightarrow{(1,5)\,(5,6)} = \langle 4, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 7 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix} = i(0) - j(0) + k(7-12)$$

$$= \langle 0, 0, -5 \rangle$$



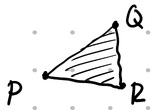
$\boxed{\text{Area} = 5}$

Find unit vector orthogonal to the plane passing through P, Q + R w/ positive first coordinate.

P(-4, 5, -1)

Q(-3, 6, 0)

R(-3, 6, 3)



$\overrightarrow{PQ} = \langle -3+4, 6-5, 0+1 \rangle = \langle 1, 1, 1 \rangle$

$\overrightarrow{PR} = \langle -3+4, 6-5, 3+1 \rangle = \langle 1, 1, 4 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = \langle 3, -3, 0 \rangle$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{9+9} = \sqrt{18}$$

$$\boxed{\hat{v} = \left\langle \frac{3}{\sqrt{18}}, \frac{-3}{\sqrt{18}}, 0 \right\rangle}$$

*Break: X:10