# Wednesday, July 6, 2022

# MATH 164 Lecture Notes

* Today's material is not on Midterm 1.

## Section 14.3: Partial Derivatives

Suppose $f(x_1, \dots, x_n)$ is a function of $n$ variables. Then the partial derivative of $f$ with respect to the variable $x_i$ is:

Not a fraction

$\frac{\partial f}{\partial x_i}$ or $f_{x_i}$ or $f_i$ or $\partial_{x_i} f$

and you get it by pretending all the other variables are constants.

"operator"

$$\frac{\partial}{\partial x}\left(2 f(x,y) + 3 g(x,y)\right) = 2\frac{\partial}{\partial x} f(x,y) + 3\frac{\partial}{\partial x} g(x,y)$$

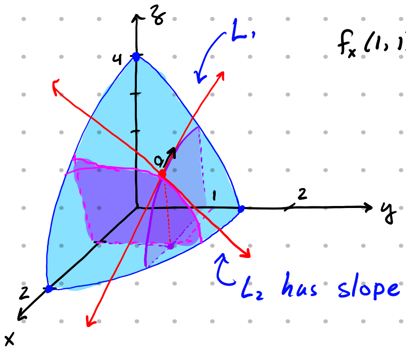
$f_x(x,y)$ $\quad$ $f_i(x_1, \dots, x_n)$

Example: $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3 \qquad \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

$$f(1,1) = 4 - 1 - 2 = 1$$

Geometry (slopes): Let $f(x,y) = 4 - x^2 - 2y^2$. Consider $f_x(1,1)$ and $f_y(1,1)$




$L_2$ has slope $= f_y(1,1)$

$$f_x(1,1) = \frac{\partial f}{\partial x}(1,1).$$

$$\frac{\partial f}{\partial x} = -2x$$

$$f_y(1,1) = -4$$

$\frac{\partial f}{\partial x}(1,1) = -2$ = slope of a line tangent to the graph of $f$ & pass throug the point $(1,1,1)$ & parallel to the $x$-axis.

Example (chain rule): $f(x,y) = \sin\left(\frac{x}{1+y}\right)$ find $f_x$ & $f_y$

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\cdot\left(\frac{1}{1+y}\right) \qquad \frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right)\cdot\left(\frac{-x}{(1+y)^2}\right)$$

Example (implicit differentiation): Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ when $x^3 + y^3 + z^3 + 6xyz = 1$

For $\frac{\partial z}{\partial x}$:

$$\frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + 6xyz\right) = 0$$

$$\Rightarrow \quad 3x^2 + 3z^2\boxed{\frac{\partial z}{\partial x}} + 6yz + 6xy\boxed{\frac{\partial z}{\partial x}} = 0 \quad \rightsquigarrow \quad \frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

For $\frac{\partial z}{\partial y}$:

$$\frac{\partial}{\partial y}\left(x^3 + y^3 + z^3 + 6xyz\right) = 0$$

$$0 + 3y^2 + 3z^2\cdot\boxed{\frac{\partial z}{\partial y}} + 6xz + 6xy\boxed{\frac{\partial z}{\partial y}} = 0$$

Example (more variables): $f(x,y,z) = e^{xy}\ln z$

$$f_y = x e^{xy} \ln z$$

$$f_x = y e^{xy} \ln(z)$$

$$f_z = \frac{e^{xy}}{z}$$

Example (Mixed Derivatives/More Derivatives): $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)$$

$$f_x = 3x^2 + 2xy^3 \qquad f_y = 3x^2y^2 - 4y$$

$$\boxed{f_{xx} = 6x + 2y^3} \qquad \boxed{f_{xy} = 6xy^2} = \boxed{f_{yx} = 6xy^2} \qquad \boxed{f_{yy} = 6x^2y}$$

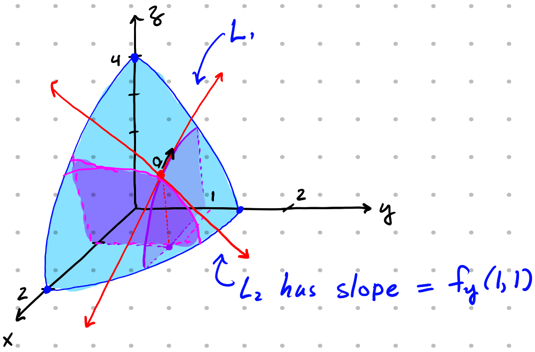
Clairaut's Theorem: Suppose $f$ is defined on a disk $D$ containing $(a,b)$. If $f_{xy}$ and $f_{yx}$ are both continuous on $D$, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

* If $f_{xy}, f_{yx}$ are not continuous then this isn't true. $f_{xy} = f_{yx}$

Example: * Find an example where $f_{xy} \neq f_{yx}$

## Section 14.4: Tangent Planes & Linear Approximations




Let $S$ be the graph of $z = f(x,y)$. Choose $P(x_0, y_0, z_0) \in S$.

Goal: Find the plane tangent to $S$ at $P(x_0, y_0, z_0)$.

We know the plane has the form

$$\boxed{A(x-x_0) + B(y-y_0) + C(z-z_0)}$$

or:

$$\boxed{z - z_0 = a(x-x_0) + b(y-y_0)}$$

* How can we find the constants $a$ & $b$?

Example: $z = \underbrace{1 - x^2 - y^2}_{f(x,y)}$   Point: $(1,1,-1)$   Find the tangent plane to the graph of $f$ at the point $(1,1,-1)$

$f_x = -2x \quad f_x(1,1) = -2$

$f_y = -2y \quad f_y(1,1) = -2$

$$z + 1 = -2(x-1) - 2(y-1)$$

$$= -2x + 2 - 2y + 2 = -2x - 2y + 4$$

$$\boxed{0 = -z - 2x - 2y + 3}$$

* Tangent planes can be thought of as "linear approximations" of surfaces/functions.

Example: Find the "linearization" of $\boxed{f(x,y) = xe^{xy}}$ at $(1,0)$

$P(1,0,1)$   $z - 1 = f_x(x-1) + f_y(y)$   $L(1.3, -.2) \approx f(1.3, -.2)$

$f_x = x \cdot \frac{\partial}{\partial x} e^{xy} + e^{xy}$   $f_x(1,0) = 1$

$f_x = xye^{xy} + e^{xy}$

$$z - 1 = x - 1 + y$$

$$\boxed{L(x,y) = x + y}$$

$f_y = x^2 e^{xy}$   $f_y(1,0) = 1$

## Section 14.5: The Chain Rule

Case 1: $z = f(x,y) \quad x = g(t) \quad y = h(t)$

temp   # of parking lots   # of cars running

$t$ = # of people living there.

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

* Proof is in the book

Example: $z = x^2 y \quad x = \sin(2t) \quad y = \cos(t)$   Find $\frac{dz}{dt}$

$\frac{\partial z}{\partial x} = 2xy \quad \frac{\partial z}{\partial y} = x^2 \quad \frac{dx}{dt} = 2\cos(2t) \quad \frac{dy}{dt} = -\sin(t)$

$$\boxed{\frac{dz}{dt} = 2\sin(2t)\cos(t)2\cos(2t) - \sin^2(2t)\sin(t)}$$

**Case 2:** $z = f(x,y) \quad x = g(s,t) \quad y = h(s,t)$

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

*Example:* $z = e^x \cdot \sin(y) \quad x = st^2 \quad y = s^2 t \quad$ Find $\frac{\partial z}{\partial s}$

$$z(s,t) = e^{st^2}\sin(s^2 t)$$

$$\frac{\partial z}{\partial x} = e^x \cdot \sin(y) \qquad \frac{\partial x}{\partial s} = t^2 \qquad \frac{\partial z}{\partial y} = e^x\cos(y) \qquad \frac{\partial y}{\partial s} = 2st$$

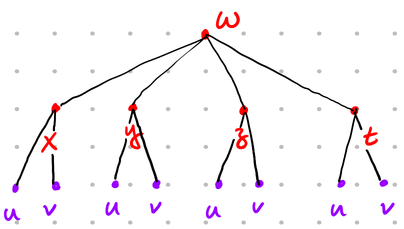
$$\frac{\partial z}{\partial s} = e^x \cdot \sin(y) \cdot t^2 + e^x\cos(y) \cdot 2st$$

**General Case:** Suppose that $f$ is a diff. function of $n$-variables $(x_1, \dots, x_n)$. Suppose that $x_j$ is a diff. function of $m$ variables $(t_1, \dots, t_m)$. Then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

*Example:* If we have $w = f(x,y,z,t)$ & $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $t = t(u,v)$

Write the chain rule for $\frac{\partial w}{\partial u}$




$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}}$$

- **Implicit Differentiation.**

$$x^2 + y^2 + z^2 = 0$$

Suppose we have something like $F(x,y) = 0$ which defines $y$ implicitly. $F(x, f(x)) = 0$

Goal: Find $\frac{dy}{dx}$

> *Implicit function Theorem 1:*
>
> $$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

*Example:* $x^3 + y^3 = 6xy \qquad F(x,y) = x^3 + y^3 - 6xy = 0 \quad$ Find $y' = \frac{dy}{dx}$

$$F_x = 3x^2 - 6y$$

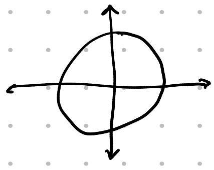
$$F_y = 3y^2 - 6x$$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2 + 6y}{3y^2 - 6x}}$$

If we have $F(x, y, z(x,y)) = 0$ we can find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

**Implicit function Theorem 2:**

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$



Example: $x^3 + y^3 + z^3 + 6xyz = 1$ Find $\frac{\partial z}{\partial x}$

$$F(x,y,z) = x^3 + y^3 + z^3 + 6xyz - 1$$

$$F_x = 3x^2 + 6yz$$

$$F_z = 3z^2 + 6xy$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{3x^2 + 6yz}{3z^2 + 6xy}}$$

**Implicit Diff.**

$x^2 + y^2 = 1$ Find $\frac{dy}{dx}$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Break: X:05

$$2x + 2y\boxed{\frac{dy}{dx}} = 0$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Practice Problems

## Section 14.4 - 14.5

Find the tangent plane at the point.

$z = x\sin(x+y)$ at $(-1, 1, 0)$ $\qquad z = a(x+1) + b(y-1)$

$a = \frac{\partial z}{\partial x} = x \cdot \cos(x+y) + \sin(x+y)$

$\frac{\partial z}{\partial x}(-1,1) = -\cos(0) + \sin(0) = -1$

$b = \frac{\partial z}{\partial y} = x\cos(x+y) \qquad \frac{\partial z}{\partial y}(-1,1) = -\cos(0) = -1$

$\Rightarrow \quad z = -x-1-y+1 = -x-y \qquad \boxed{z = -x-y}$

Find $\partial y/\partial x$

method 1: directly

$y\cos x = x^2+y^2$

LHS $\frac{\partial}{\partial x}(y\cos x) = \boxed{\frac{\partial y}{\partial x}}\cdot\cos x - y\sin(x)$

RHS $\frac{\partial}{\partial x}(x^2+y^2) = 2x + 2y\cdot\boxed{\frac{\partial y}{\partial x}}$

$\boxed{\frac{\partial y}{\partial x}}\cdot\cos x - y\sin(x) = 2x + 2y\cdot\boxed{\frac{\partial y}{\partial x}}$

$\boxed{\frac{\partial y}{\partial x}}\cdot\cos x - 2y\cdot\boxed{\frac{\partial y}{\partial x}} = 2x + y\sin(x)$

$\frac{\partial y}{\partial x}(\cos x - 2y) = 2x + y\sin(x)$

$$\boxed{\frac{\partial y}{\partial x} = \frac{2x + y\sin(x)}{\cos x - 2y}}$$

method 2: using formula.

$F(x,y) = y\cos x - x^2 - y^2$

$F_x = -y\sin x - 2x$

$F_y = \cos x - 2y$

$$\boxed{\frac{\partial y}{\partial x} = \frac{y\sin x + 2x}{\cos x - 2y}}$$

**15.** Suppose $f$ is a differentiable function of $x$ and $y$, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

| | $f$ | $g$ | $f_x$ | $f_y$ |
|---|---|---|---|---|
| (0, 0) | 3 | 6 | 4 | 8 |
| (1, 2) | 6 | 3 | 2 | 5 |

$g(u,v) = f(\underbrace{e^u + \sin v}_{x(u,v)}, \underbrace{e^u + \cos v}_{y(u,v)})$ diff

Find $\frac{\partial g}{\partial u}(0,0)$ and $\frac{\partial g}{\partial v}(0,0)$

Write the chain rule for $g(u,v)$

$$\boxed{\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}}$$

$\frac{\partial x}{\partial u} = e^u \qquad \frac{\partial y}{\partial u} = e^u$

$\frac{\partial x}{\partial u}(0,0) = 1 \qquad \frac{\partial y}{\partial u}(0,0) = 1$

$\frac{\partial g}{\partial u}(0,0) =$

When $(u,v) = (0,0)$, $x(0,0) = 1 \qquad y(0,0) = 2$

$\frac{\partial g}{\partial u}(0,0) = \frac{\partial f}{\partial x}(1,2)\cdot\frac{\partial x}{\partial u}(0,0) + \frac{\partial f}{\partial y}(1,2)\frac{\partial y}{\partial u}(0,0)$

$= (2)(1) + (5)(1) = \boxed{7}$

29) From 14-3 Find $F_x$ & $F_y$

$$F(x,y) = \int_y^x \cos(e^t)\,dt$$

$$= \underbrace{\int_a^x \cos(e^t)\,dt}_{G(x)} - \underbrace{\int_a^y \cos(e^t)\,dt}_{H(y)}$$

$$\frac{\partial F}{\partial x} = \frac{dG}{dx} = \cos(e^x) \qquad \frac{\partial F}{\partial y} = -\frac{dH}{dy} = -\cos(e^y)$$

FTC

$$F(x) = \int_a^x f(t)\,dt$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

## Participation Points Questions

1. Find $\frac{\partial f}{\partial x}$ when $f(x,y,z) = \cos(x)y^2 + zx$

2. Suppose $x = r\cos(t)$, $y = r\sin(t)$, and $f(x,y) = x \cdot y$. Find $\frac{\partial f}{\partial t}$.