Tuesday, July 5, 2022

MATH 164 Lecture Notes

Chapter 14: Partial Derivatives

Section 14.1: Functions of several variables

Definition: A function of n variables is a rule that assigns to each ordered tuple of real numbers $(x_1, \dots, x_n)$ in a set D a unique real number denoted by $f(x_1, \dots, x_n)$. The set D is called the domain of f and its range is the set

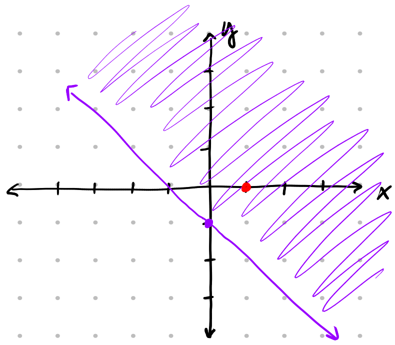
dependent

$$\text{rang}(f) = \{ f(x_1, \dots, x_n) \mid (x_1, \dots, x_n) \in D \}$$

We may write $z = f(x_1, \dots, x_n)$. independent variables

Example: Sketch the domain:

$$f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$$



$\sqrt{x+y+1} \geq 0$

$x+y+1 \geq 0$

$x+y \geq -1$

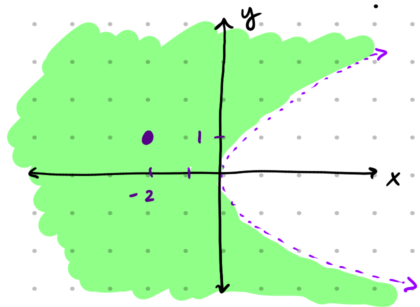
$y \geq -x-1$

$f(x,y) = x \ln(y^2 - x)$

$D(f) = \{ (x,y) \in \mathbb{R}^2 : y^2 - x > 0 \}$

$y^2 - x > 0$

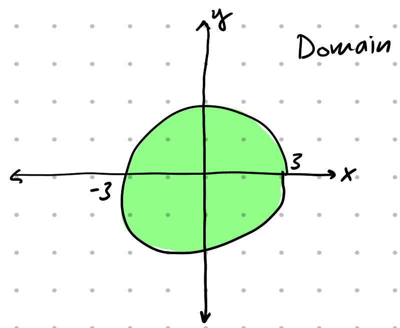
$y^2 > x$



Example: Find the domain and range of $f(x,y) = \sqrt{9 - x^2 - y^2}$

Domain: $D(f) = \{ (x,y) : 9 - x^2 - y^2 \geq 0 \Rightarrow 9 \geq x^2 + y^2 \}$

Range: $R(f) = \{ k \in \mathbb{R} : \sqrt{9-x^2-y^2} = k \iff 9 - x^2 - y^2 = k^2 \} = [0,3]$



$f(x,y,z,w)$ $\quad (x,y,z,w) \in \mathbb{R}^4$

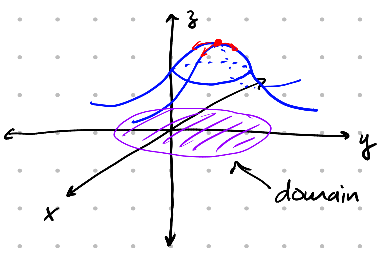
- Graphs

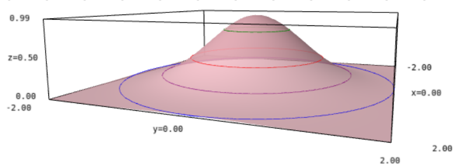
Definition: If f is a function of two variables with domain D, then the graph of f is:

$$\text{graph}(f) = \{ (x,y,z) \in \mathbb{R}^3 : z = f(x,y), (x,y) \in D \}$$

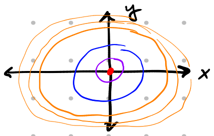
* graphs are a special case of surfaces.

Example: $f(x,y) = e^{-x^2 - y^2}$

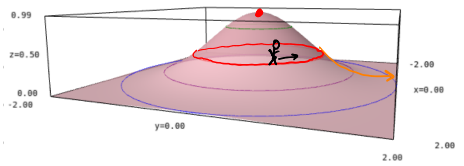




Level Curves



Definition: The level curves of a function f are curves of the form $f(x,y)=k$ where k is a constant in the range of f. (think traces).



$f(x,y) = e^{-x^2-y^2}$

$\cancel{f(x,y)} = 0$?

negative number $-k$

$e^{-x^2-y^2} = 1$

$e^{-x^2-y^2} = .5 \iff -x^2-y^2 = \ln(.5)$

$e^r = 1 \quad -x^2-y^2 = 0 \iff (x,y) = (0,0)$

$\boxed{x^2+y^2=k}$

When we have 3 or more variables, we can no longer visualize the graphs but we can still get an idea of how the function behaves.

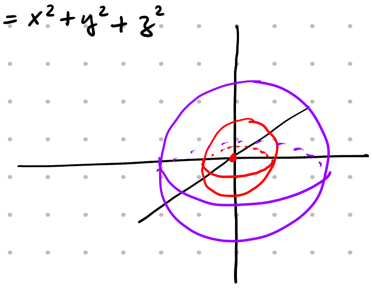
Example: Find the domain of $f(x,y,z) = \ln(z-y) + xy\sin(z)$

Domain: $\{(x,y,z) \in \mathbb{R}^3 : z-y \geq 0\}$

Example: Find the level surfaces of $f(x,y,z) = x^2+y^2+z^2$

$x^2+y^2+z^2 = 1 \rightsquigarrow$ sphere

$x^2+y^2+z^2 = 0 \rightsquigarrow$ point



* * Section 14.2: Limits and Continuity * *

Example: Compare the following:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Let $x^2+y^2 = r$ $\quad \lim_{r\to 0} \frac{\sin(r)}{r} = 1$

* You must prove that the limit is the same no matter which path you take.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Let $y=0$ $\quad \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

Let $x=0$ $\quad \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

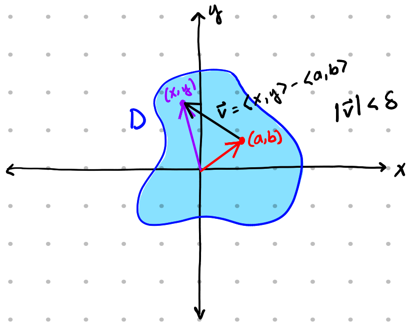
$1 \neq -1 \implies$ limit DNE

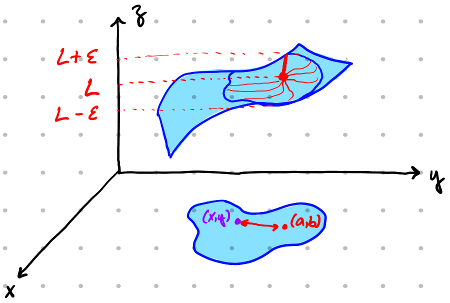
What can you say about how these functions behave at the origin?

Definition: Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) goes to (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

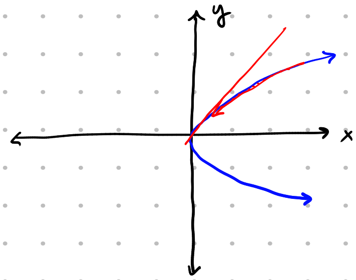
if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.





$y = k$

* Showing a limit DNE is usually a lot easier than showing it does.



Limit Strategies

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ Show DNE

Let $x = 0$ $\displaystyle\lim_{(0,y)\to(0,0)} \frac{0}{y^4} = 0$

Let $y = 0$ $\displaystyle\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$

Let $x = y^2$ $\displaystyle\lim_{y\to 0} \frac{\cancel{y^4}}{2\cancel{y^4}} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$ ✓ DNE

Let $x = my$ $\displaystyle\lim_{y\to 0} \frac{my^3}{m^2y^2+y^4} = \lim_{y\to 0} \frac{my}{m^2+y^2} = 0$

2. Radial Symmetry ⇝ limit exists

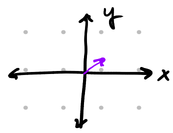
$\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ plug in $x^2+y^2 = r$

* substitution.

3. "Squeeze" ⇝ limit exists

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$ then $\left|\dfrac{3x^2y}{x^2+y^2}\right| < \varepsilon$



Observation: $x^2 \le x^2+y^2 \Rightarrow \dfrac{x^2}{x^2+y^2} \le 1$

Assume that $\sqrt{x^2+y^2} < \delta \overset{?}{\Rightarrow} \left|\dfrac{3x^2y}{x^2+y^2}\right| < \varepsilon \iff \dfrac{3x^2|y|}{x^2+y^2} < \varepsilon$

$$\frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2} \le 3\cdot\delta$$

* Let $\varepsilon = 3\cdot\delta$. $|f(x,y) - L| \le \varepsilon$

$$\lim_{(x,y)\to(0,0)} \left| \frac{3x^2y}{x^2+y^2} \right| = \lim_{(x,y)\to(0,0)} \left| 3 \cdot y \frac{x^2}{x^2+y^2} \right| \le \lim_{(x,y)\to(0,0)} |3y| = 0$$

$\Rightarrow$ limit must be zero.

☐ Continuity

Definition: A function of two variables is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. We say a function f is continuous if it's continuous at all the points in its domain D.

Example: Is f continuous?

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Where is $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous? $\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$

Examples + Practice Problems

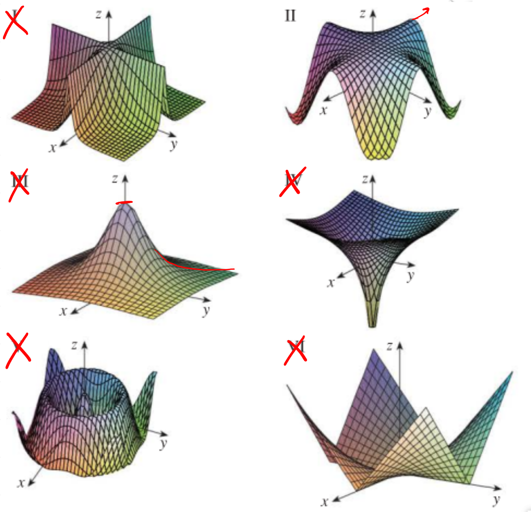
Section 14.1

32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = \dfrac{1}{1 + (x^2 + y^2)}$ (b) $f(x, y) = \dfrac{1}{1 + x^2y^2}$

(c) $f(x, y) = \ln(x^2 + y^2)$ (d) $f(x, y) = \cos\sqrt{x^2 + y^2}$

(e) $f(x, y) = |xy|$ (f) $f(x, y) = \cos(xy)$



a) III

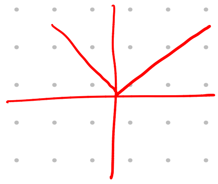
b) I

c) IV

d) V

e) VI

f) II



67–70 Describe the level surfaces of the function.

67. $f(x, y, z) = x + 3y + 5z$

68. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

69. $f(x, y, z) = y^2 + z^2$

70. $f(x, y, z) = x^2 - y^2 - z^2$

67) $x + 3y + 5z = k$

The level surfaces are all planes with normal vector $\vec{n} = \langle 1, 3, 5 \rangle$

68) $x^2 + 3y^2 + 5z^2 = k$

ellipsoids

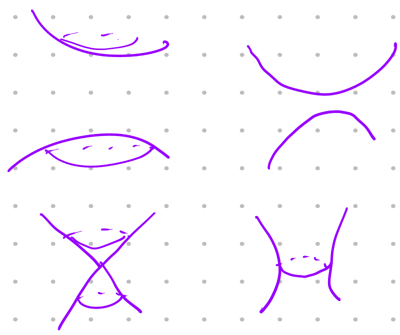
69) $y^2 + z^2 = k$

cylinders

70) $x^2 - y^2 - z^2 = k$ Hyperboloid of two sheets

$x^2 - y^2 - z^2 = 0 \rightsquigarrow x^2 = y^2 + z^2 \rightsquigarrow$ cone

$-x^2 + y^2 + z^2 = k$ Hyperboloid of one sheet



Section 14.2

5–22 Find the limit, if it exists, or show that the limit does not exist.

5. $\displaystyle\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$

6. $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{x^2y + xy^2}{x^2 - y^2}$

7. $\displaystyle\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x - y)$

8. $\displaystyle\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}}$

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$

10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4 + y^4}$

11. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4 + y^4}$

12. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$

13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + xy + y^2}$

15. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2\cos y}{x^2 + y^4}$

16. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4 + y^4}$

17. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

18. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$

19. $\displaystyle\lim_{(x,y,z)\to(\pi,0,1/3)} e^{y^2}\tan(xz)$

20. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{x^2 + y^2 + z^2}$

5) $\displaystyle\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$

$9 \cdot 8 - 16$

9) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$

let $x = 0$ $\displaystyle\lim_{(0,y)\to(0,0)} \frac{-4y^2}{2y^2} = -2$

let $y = 0$ $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x^4}{x^2} = 0$ $\quad 0 \neq -2$ DNE

13) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$\displaystyle\left|\frac{xy}{\sqrt{x^2+y^2}}\right|$ $\qquad \displaystyle\frac{|x|}{\sqrt{x^2+y^2}} \leq 1$

$\displaystyle = \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}} \leq |y|$ $\qquad \displaystyle\frac{|x|}{\sqrt{x^2}} = 1$

$\displaystyle\lim_{y\to 0} |y| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = 0$

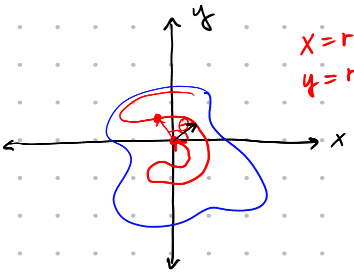
✓

39–41 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

39. $\displaystyle \lim_{(x, y)\to(0, 0)} \frac{x^3 + y^3}{x^2 + y^2}$

40. $\displaystyle \lim_{(x, y)\to(0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

41. $\displaystyle \lim_{(x, y)\to(0, 0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$



$x = r\cos\theta$
$y = r\sin\theta$

$r \to 0$

$\cancel{\theta \to 0}$

$\cancel{\lim_{(r,\theta)\to(0,0)}}$

39) $x = r\cos\theta$

$y = r\sin\theta$

$$\frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$\lim_{(r,\theta)\to(0,\theta)} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{(r,\theta)\to(0,\theta)} \frac{\cancel{r^3}\cos^3\theta + \cancel{r^3}\sin^3\theta}{\cancel{r^2}}$$

$$= \lim_{(r,\theta)\to(0,\theta)} \left(r\cos^3\theta + r\sin^3\theta\right) = 0$$

40) $\displaystyle \lim_{r\to 0^+} r^2 \ln(r^2) = 0$

Participation Points Questions

1. What are the level surfaces of $f(x,y,z) = x^2 + y^2$?

2. What is the domain of $f(x,y) = \dfrac{x+y}{x-1}$?