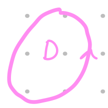
Thursday, July 28, 2022

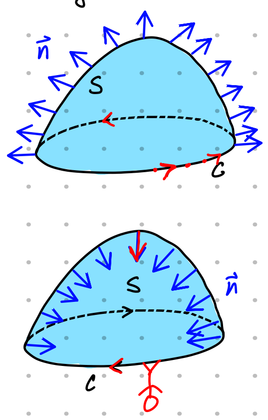
MATH 164 Lecture Notes



$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\vec{\nabla}\times\vec{F})\,dA$

Section 16.8: Stokes' Theorem

Positively oriented ∂S:



S with a normal vector field.

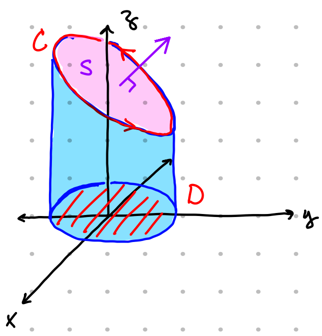
Theorem: Let S be an oriented, piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region of $\mathbb{R}^3$ containing S. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

D dA

Example: Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = \langle -y^2, x, z^2\rangle$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$



$$\text{Curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y\rangle$$

- Parameterize the plane $y+z=2$

$$\vec{r}(x,y) = \langle x, y, 2-y\rangle$$

$$\iint_D \langle 0,0,1+2y\rangle\cdot d\vec{S} = \iint_D (1+2y)\,dA$$

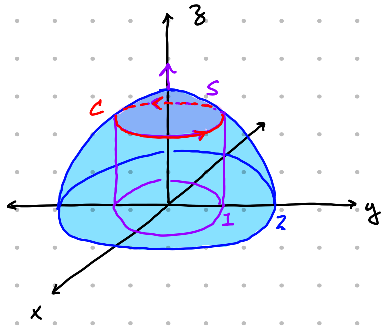
$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{1}{2}+\tfrac{2}{3}\sin\theta\right)d\theta$$

$$= \tfrac{1}{2}\theta - \tfrac{2}{3}\cos\theta\,\Big|_0^{2\pi} = \pi$$

$\vec{r}(x,y) = \langle x, y, 2-y\rangle$ $\quad \vec{r}_x\times\vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, -1, 1\rangle$

$\langle 0,0,1+2y\rangle\cdot\langle 0,-1,1\rangle = (1+2y)$

Example: Use Stokes' Theorem to evaluate $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ where $\vec{F} = \langle xz, yz, xy \rangle$ & S is the part of the sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2+y^2=1$ & above the xy-plane.



- Find C: $1 + z^2 = 4 \Rightarrow z = \sqrt{3}$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

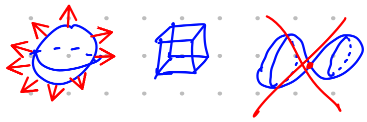
- Parameterize C : $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$ $\quad 0 \le t \le 2\pi$

$\vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_0^{2\pi} \langle \cos(t)\sqrt{3}, \sin(t)\sqrt{3}, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\sin(t)\cos(t) + \sqrt{3}\sin(t)\cos(t) + 0 \right) dt = 0$$

Section 16.9: The Divergence Theorem



Theorem: Let E be a simple, solid region in $\mathbb{R}^3$ and let $S = \partial E$ with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region containing E. Then

$\vec{\nabla}\cdot\vec{F}$ $\qquad \vec{\nabla} = \langle \partial_x, \partial_y, \partial_z \rangle$

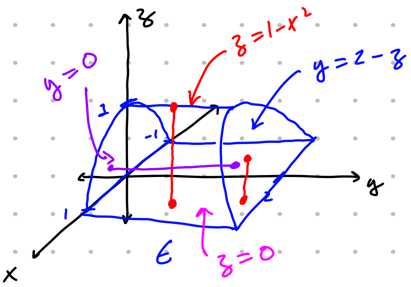
$$\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{Flux}} = \iiint_E \operatorname{div}\vec{F}\, dV$$

Example: Find the flux of $\vec{F} = \langle z, y, x\rangle$ over the unit sphere $x^2+y^2+z^2 \le 1$

$$\vec{\nabla}\cdot\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle \cdot \langle z, y, x\rangle = \partial_x z + \partial_y y + \partial_z x = 1$$

$$\iiint_E dV = \text{Volume of the unit sphere: } \frac{4\pi}{3}$$

Example: Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy)\rangle$ & S is the surface of the region E bounded by the parabolic cylinder $z = 1-x^2$ & the planes $z=0, y=0$ & $y+z=2$.

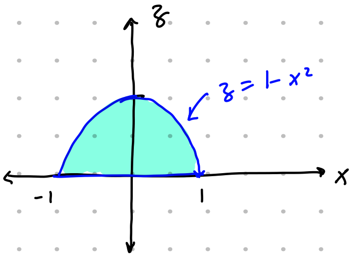


$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV}$$

$$\operatorname{div}\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle \cdot \langle xy,\ y^2 + e^{xz^2},\ \sin(xy)\rangle$$

$$= y + 2y + 0 = 3y$$

$$\iiint_E 3y\, dV = 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\, dy\, dz\, dx$$



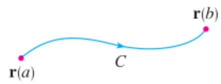
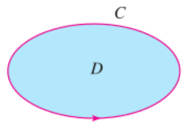
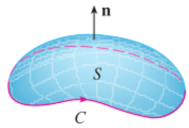
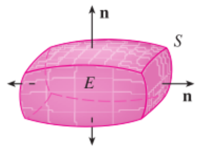
$$= 3\int_0^1\int_0^{1-x^2} \frac{1}{2}(2-z)^2\, dz\, dx$$

$$(2-z)(2-z) = 4 - 4z + z^2$$

$$= \frac{3}{2}\int_0^1 4z - 2z^2 + \frac{1}{3}z^3 \Big|_0^{1-x^2} dx$$

$$= \frac{3}{2}\int_0^1 \left(4(1-x^2) - 2(1-x^2)^2 + \frac{1}{3}(1-x^2)^3\right) dx$$

Section 16.10: Summary

Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

Part A of the Final:

- Green's Theorem
- Curl + Divergence
- parametric surfaces
- surface integrals
- Stokes' Theorem + Div theorem.

The Final will last 3 hours.

~1.5 hours for Part A (5 questions)

~1.5 hours for Part B (7 questions)

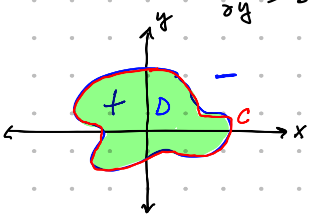
$\vec{F} \cdot d\vec{r}$

$F = \langle y^3 - y,\ 2x^3 \rangle$

Green's Theorem Example: Find the positively oriented simple closed curve C that maximizes the integral $\int_C (y^3 - y)\,dx - 2x^3\,dy$.

$$\int_C \overset{P}{(y^3-y)}\,dx - \overset{Q}{2x^3}\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\frac{\partial P}{\partial y} = 3y^2 - 1 \qquad \frac{\partial Q}{\partial x} - 6x^2 \quad \Rightarrow \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -6x^2 - 3y^2 + 1 = f(x,y)$$



$$f(x,y) = 0 \quad \Rightarrow \quad -6x^2 - 3y^2 + 1 = 0 \quad \Leftrightarrow \quad \boxed{1 = 6x^2 + 3y^2}$$

$x = \frac{1}{\sqrt{6}}\cos(t) \quad y = \frac{1}{\sqrt{3}}\sin(t)$ Equation of an ellipse!

$f(0,0) = 1 > 0$ Answere is $\vec{r}(t) = \left\langle \frac{1}{\sqrt{6}}\cos(t), \frac{1}{\sqrt{3}}\sin(t) \right\rangle$

7–10 Evaluate the double integral.

$x=0 \quad y=0$

$x=4 \quad y=\sqrt{x} \rightsquigarrow x=y^2$

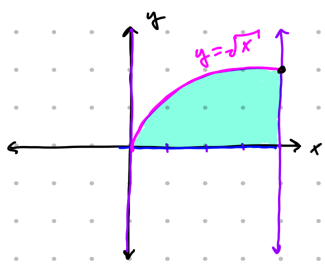
7. $\iint_D \frac{y}{x^2+1}\,dA, \quad D=\{(x,y) \mid 0 \leqslant x \leqslant 4, 0 \leqslant y \leqslant \sqrt{x}\}$

8. $\iint_D (2x+y)\,dA, \quad D=\{(x,y) \mid 1 \leqslant y \leqslant 2, y-1 \leqslant x \leqslant 1\}$

9. $\iint_D e^{-y^2}\,dA, \quad D=\{(x,y) \mid 0 \leqslant y \leqslant 3, 0 \leqslant x \leqslant y\}$

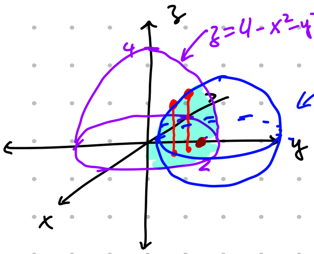
10. $\iint_D y\sqrt{x^2-y^2}\,dA, \quad D=\{(x,y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant x\}$

7)



$$\int_0^4\int_0^{\sqrt{x}} \frac{y}{x^2+1}\,dy\,dx = \int_0^4 \frac{1}{2}\cdot\frac{x}{x^2+1}\,dx$$

Example: Find the volume under the paraboloid $-x^2-y^2+4=z$ + inside the sphere w/ center $(0,2,0)$ + radius 1. + above the xy-plane.

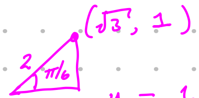


$z=\sqrt{1-x^2-(y-2)^2}$

$$x^2+(y-2)^2+z^2=4$$

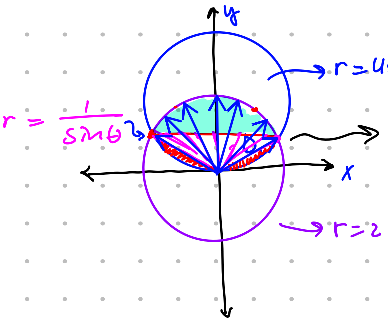
$$x^2+(y-2)^2=1$$

$$\int_{\pi/6}^{5\pi/6}\int_{1/\sin\theta}^{2}\int_0^{4-x^2-y^2} dz\,r\,dr\,d\theta + \int_{\pi/6}^{5\pi/6}\int_{D_{11}}^{1/\sin\theta}\int_0^{\sqrt{1-x^2-(y-2)^2}} dz\;r\,dr\,d\theta + 2\int_0^{\pi/6}\int_0^{4\sin\theta}\int_0^{\sqrt{1-x^2-(y-2)^2}} dz$$



$y=1$

$r\sin\theta=1$



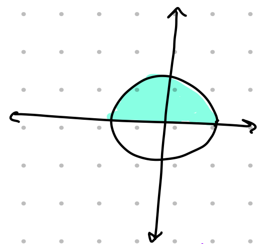
$\frac{1}{2}=\sin\theta \Rightarrow \theta=\frac{\pi}{6}$ + $\frac{5\pi}{6}$

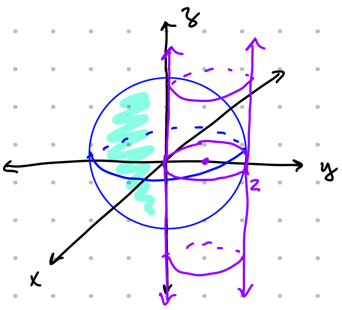
$$r^2\cos^2\theta+(r\sin\theta-2)^2=4$$
$$r^2\cos^2\theta++r^2\sin^2\theta-4r\sin\theta+4=4$$
$$r^2-4r\sin\theta+4=4$$
$$r(r-4\sin\theta)=0$$

Problem 3

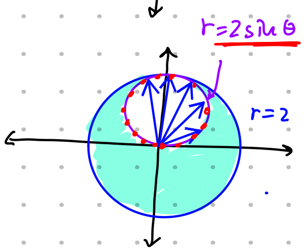
Set up but **do not evaluate** the integral representing the volume of the solid contained **inside** the sphere $4=x^2+y^2+z^2$ and **outside** the cylinder $x^2+(y-1)^2=1$

Hint: It might be helpful to think of this volume as a *double integral* although this is not strictly necessary.





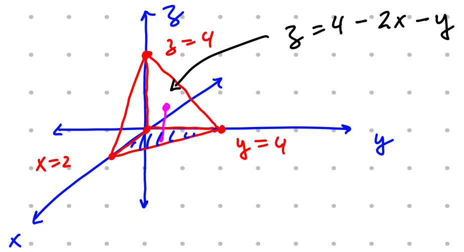
$$r^2\cos^2\theta+(r\sin\theta-1)^2=1$$
$$r^2\cos^2\theta+r^2\sin^2\theta-2r\sin\theta+1=1$$
$$r^2-2r\sin\theta=0$$
$$r=2\sin\theta$$
$$4=r^2+z^2 \rightsquigarrow z=\sqrt{4-r^2}$$

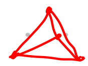


$$2\int_0^{2\pi}\int_0^2 \sqrt{4-r^2}\,r\,dr\,d\theta - 2\int_0^{\pi}\int_0^{2\sin\theta}\sqrt{4-r^2}\,r\,dr\,d\theta$$

19–22 Use a triple integral to find the volume of the given solid.

19. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$





$$\int_0^2 \int_0^{-2x+4} \int_0^{4-2x-y} dz\, dy\, dx$$