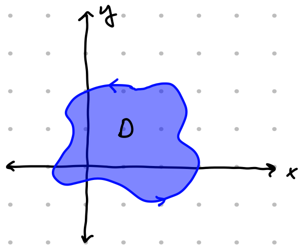
Wednesday July 27, 2022

MATH 164 Lecture Notes



Section 16.5 Continued

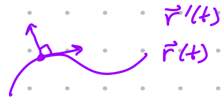
Green's Theorem: $\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \boxed{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} dA$

Curl: $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = (\partial_y R - \partial_z Q)\vec{i} - (\partial_x R - \partial_z P)\vec{j} + \boxed{(\partial_x Q - \partial_y P)}\,k$

Divergence: $\vec{\nabla}\cdot\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle \cdot \langle P, Q, R\rangle = \partial_x P + \partial_y Q + \partial_z R$

Vector Forms of Green's Theorem

① $\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\vec{\nabla}\times\vec{F})\cdot \vec{k}\, dA}$



② Suppose C is given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$.

Recall that the unit tangent vector for $\vec{r}(t)$ is : $\vec{T}(t) = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$

And the unit normal vector for $\vec{r}(t)$ is : $\vec{n}(t) = \frac{y'(t)}{|\vec{r}'(t)|}\vec{i} - \frac{x'(t)}{|\vec{r}'(t)|}\vec{j}$

We have:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_a^b (\vec{F}\cdot\vec{n})(t)\,|\vec{r}'(t)|\,dt$$

$$= \int_a^b \left[\frac{P(x(t),y(t))\,y'(t)}{|\vec{r}'(t)|} - \frac{Q(x(t),y(t))\,x'(t)}{|\vec{r}'(t)|}\right] |\vec{r}'(t)|\,dt$$

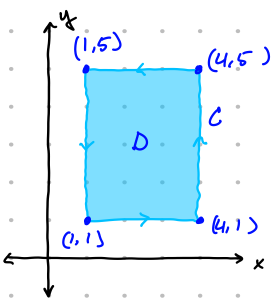
$$= \int_a^b \left[P(x(t),y(t))\,y'(t) - Q(x(t),y(t))\,x'(t)\right] dt$$

$$= \int_C P\,dy - Q\,dx = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)}_{\vec{\nabla}\cdot\vec{F}} dA$$

$\Rightarrow$ $\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \vec{\nabla}\cdot\vec{F}\,dA}$

* See problem 38 in the book for a physics application.

Example: Calculate the line integral $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle x^2y,\, y-3\rangle$ & C is the rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$, positively oriented.



$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

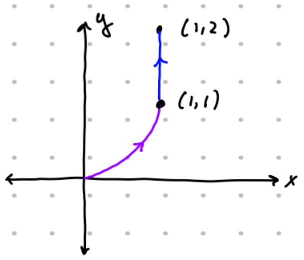
$P = x^2y \Rightarrow \frac{\partial P}{\partial y} = x^2$

$Q = y-3 \Rightarrow \frac{\partial Q}{\partial x} = 0$

$\int_1^4 \int_1^5 -x^2\,dy\,dx = \int_1^4 -x\,4\,dx = \frac{-4}{2}(16-1)$

Find my mistake:

Example: Evaluate $\int_C 2x\,ds$ where $C = C_1 \cup C_2$, $C_1 = \{(x,y)\in \mathbb{R}^2 : y = x^2,\ 0 \le x,y \le 1\}$ +
$C_2 = \{(x,y)\in\mathbb{R}^2 :\ x = 1,\ 1 \le y \le 2\}$



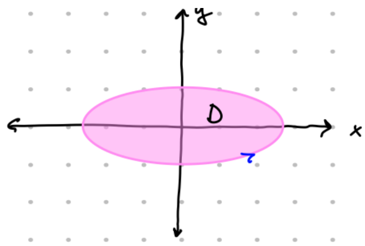
• $\vec{r}_1(x) = \langle x, x^2\rangle \quad 0 \le x \le 1 \qquad \vec{r}\,'(x) = \langle 1, 2x\rangle \qquad |\vec{r}\,'(x)| = \sqrt{1+4x^2}\,dx$

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{1+4x^2}\,dx$$

• $\vec{r}_2(y) = \langle 1, y\rangle \quad \overset{1}{\cancel{x}} \le y \le \overset{2}{\cancel{x}} \qquad \vec{r}_2\,'(y) = \langle 0, 1\rangle \qquad |\vec{r}\,'(y)| = \sqrt{1}\,dy = dy$

$$\int_{C_2} 2x\,ds = \int_{\cancel{x}\,1}^{\cancel{x}\,2} 2\,dy \qquad \int_C 2x\,ds = \int_0^1 2x\sqrt{1+4x^2}\,dx + \int_{\cancel{x}\,1}^{\cancel{x}\,2} 2\,dy.$$

Example: Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



• Green's Theorem says:

$$\text{Area}(D) = \iint_D dA = \iint_D \Big(\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=1}\Big)\,dA = \int_C P\,dx + Q\,dy$$

$x(t) = a\cos(t)$
$y(t) = b\sin(t)$

$x^2 = a^2\cos^2 t$
$y^2 = b^2\sin^2(t)$

• So we want $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$. For example, we can let $Q = x$ + $P = 0$.

$$\text{Area} = \int_0^{2\pi} ab\cos^2(t)\,dt \qquad dy = b\cos(t)\,dt$$

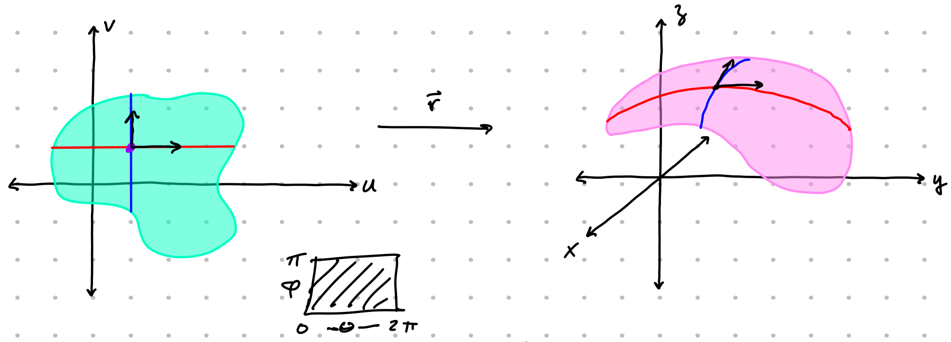
$$= \frac{ab}{2}\left(t + \sin(t)\cos(t)\right)\Big|_0^{2\pi} = \frac{ab}{2}(2\pi) = ab\pi$$

if $a = b$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2+y^2}{a^2} = 1$

$\rightsquigarrow x^2 + y^2 = a^2$

Section 16.6: Parametric Surfaces

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

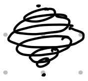


$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Guess what the surface will look like

* see what curve you get when u or v is constant.

Spherical coordinates:
$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$

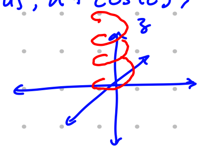


1. $\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$

2. $\vec{r}(u,v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u + \cos v \rangle$

When $v = C$ $\quad \vec{r}(u) = \langle (2+C)\cos(u), (2+C)\sin(u), u + \cos(C) \rangle$

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

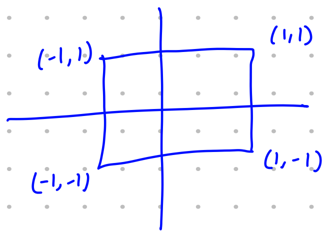


Parameterize the paraboloid $z = x^2 + y^2$

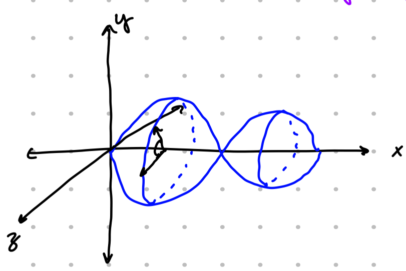
$y = x^2$

$\vec{r}(t) = \langle t, t^2 \rangle$ $\leftarrow y(t)$

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
$= \langle u, v, u^2 + v^2 \rangle$



Parameterize the surface you get when you rotate $y = \sin(x)$, $0 \le x \le 2\pi$, around the x-axis.

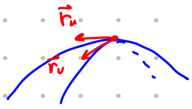


$\vec{r}(x,\theta) = \langle x, \cos(\theta)\cdot\sin(x), \sin\theta \sin(x) \rangle$

In general: If $y = f(x)$, the parametric equation for the surface you get by rotating around the x-axis:

$\vec{r}(x,\theta) = \langle x, f(x)\cdot\cos\theta, f(x)\cdot\sin\theta \rangle$

Tangent planes



- If $\vec{r}(u,v)$ is a parametric surface, $\vec{r}_u \times \vec{r}_v$ is normal at each point.
- Plane equation $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

Example: Find the tangent plane to $\vec{r}(u,v) = \langle u^2, v^2, u+2v \rangle$ at $(1,1,3)$

- $\vec{r}_u = \frac{\partial}{\partial u} \langle u^2, v^2, u+2v \rangle = \frac{\partial x}{\partial u} \vec{i} + \frac{\partial y}{\partial u} \vec{j} + \frac{\partial z}{\partial u} \vec{k} = \langle 2u, 0, 1 \rangle$

- $\vec{r}_v = \frac{\partial}{\partial v} \langle u^2, v^2, u+2v \rangle = \langle 0, 2v, 2 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = \langle -2v, -4u, 4uv \rangle$$

$\vec{r}(u,v) = \langle 1, 1, 3 \rangle$

$u^2 = 1 \quad v^2 = 1 \quad u + 2v = 3 \qquad \begin{matrix} u = 1 \\ v = 1 \end{matrix} \; \text{or} \; \begin{matrix} u = -1 \\ v = 2 \end{matrix}$

$\vec{n} = \langle -2, -4, 4 \rangle$

$$\boxed{\langle -2, -4, 4 \rangle \cdot \langle x-1, y-1, z-3 \rangle = 0}$$

Surface Area

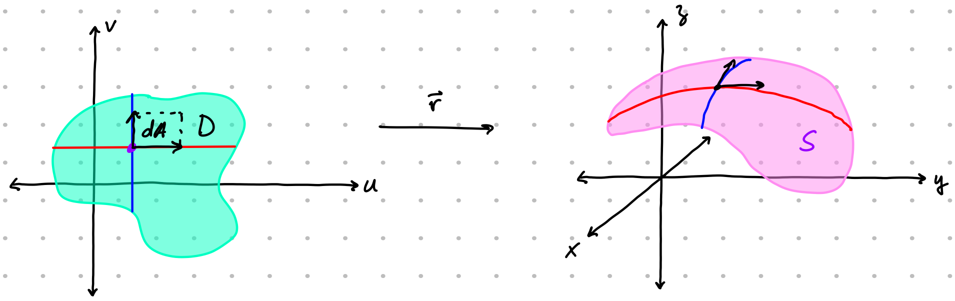
6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA \qquad dS = |\vec{r}_u \times \vec{r}_v| \cdot du\,dv$$

where $\quad \mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$



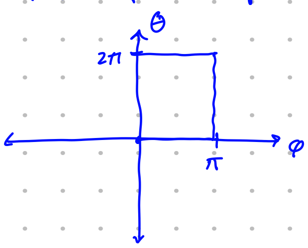
Example: Find the surface area of a sphere of radius a. $4\pi a^2$

$\vec{r}(\varphi,\theta) = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi \rangle$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x/\partial\varphi & \partial y/\partial\varphi & \partial z/\partial\varphi \\ \partial x/\partial\theta & \partial y/\partial\theta & \partial z/\partial\theta \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \langle a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\theta \rangle$$

$|\vec{r}_\varphi \times \vec{r}_\theta| = a^2\sin\varphi$



$$A(S) = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi \; d\varphi\, d\theta = 2\pi a^2 \int_0^{\pi} \sin\varphi\, d\varphi$$

$$= -2\pi a^2 \left(\cos(\pi) - \cos(0)\right) = -2\pi a^2(-2) = 4\pi a^2$$

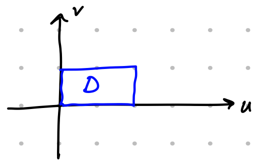
If $z = x^2 + y^2$ describes a surfface S:

$$\boxed{A(S) = \iint_D \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\, dA}$$

Section 16.7: Surface Integrals

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$$

- Assume D is a rectangle.



Integrate $f(x,y,z)$ over the surface S given by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA}$$

$$\int_C f(x,y)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

- find $\vec{r}_u$ + $\vec{r}_v$ calculate $|\vec{r}_u \times \vec{r}_v|$

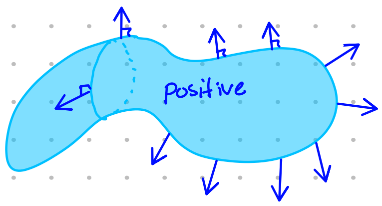
Example: Find $\iint_S x^2\, dS$ where S is the unit sphere.

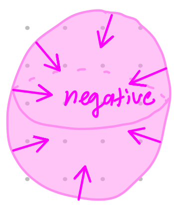
$\vec{r}(\varphi,\theta) = \langle \sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi \rangle$ $\quad |\vec{r}_\varphi \times \vec{r}_\theta| = \sin\varphi$

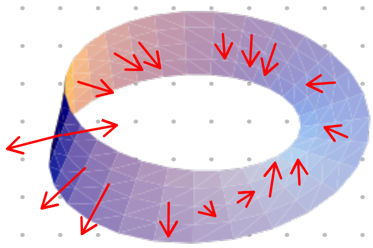
$$\int_0^{2\pi}\int_0^{\pi} \sin^3\varphi\cos^2\theta\, d\varphi\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{\pi} \sin^3\varphi\, d\varphi = \frac{4\pi}{3}$$

- Integrate Vector Fields over surfaces.

Closed, bounded surfaces.







Non-orientable

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \langle dx, dy \rangle$$
$$P\,dx + Q\,dy$$

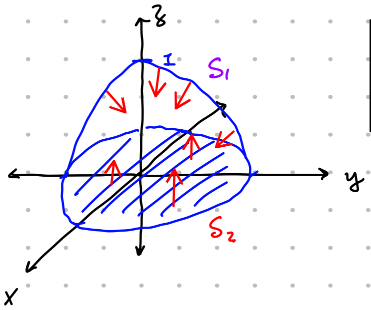
Definition: If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal $\vec{n}$ then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}}\, \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA}$$

Examples: Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle y, x, z\rangle$ & S is the boundary of the solid region enclosed by $z = 1-x^2-y^2$ & $z=0$.



$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA}$$ Flux.

1. Parameterize S_1

$\vec{r}(x,y) = \langle x, y, 1-x^2-y^2\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \vec{r}_x = \langle 1, & 0, & -2x\rangle \\ \vec{r}_y = \langle 0, & 1, & -2y\rangle \end{vmatrix} = \langle 2x, 2y, 1\rangle$$

- $\iint_{S_1} \vec{F}\cdot d\vec{S}$ $\quad \vec{F}\cdot(\vec{r}_u\times\vec{r}_v) = \langle y, x, 1-x^2-y^2\rangle\cdot\langle 2x, 2y, 1\rangle = 2xy + 2xy + 1-x^2-y^2$
$\qquad 4xy + 1-x^2-y^2$

$$\int_0^{2\pi}\int_0^1 r(4r^2\cos\theta\sin\theta + 1 - r^2)\,dr\,d\theta = \frac{\pi}{2}$$

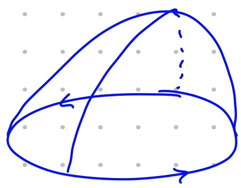
- Integral over S_2: Parameterize $z=0$ $\quad \vec{r}(x,y) = \langle x, y, 0\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \vec{r}_x = \langle 1, & 0, & 0\rangle \\ \vec{r}_y = \langle 0, & 1, & 0\rangle \end{vmatrix} = \langle 0, 0, 1\rangle$$ use $\langle 0, 0, -1\rangle$ so it's positively oriented.

$\vec{F}(\vec{r}(x,y)) = \langle y, x, 0\rangle \qquad \vec{F}\cdot(\vec{r}_u\times\vec{r}_v) = 0$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

Tomorrow: Stokes' Theorem



$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$$

Participation Points

1. Write a parametric (vector) equation describing the cylinder $x^2+y^2=1$ in $\mathbb{R}^3$