Tuesday, July 26, 2022
MATH 164 Lecture Notes

Quick Line Integral Review

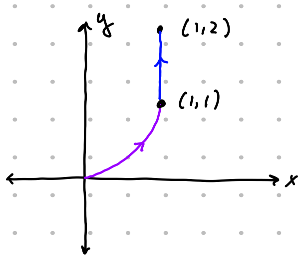
$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

- $C = \{(x,y) \in \mathbb{R}^2 : x = x(t),\ y = y(t)\}$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$ds = |\vec{r}\,'(t)|\,dt \quad a \le t \le b$

Example: Evaluate $\int_C 2x\,ds$ where $C = C_1 \cup C_2$, $C_1 = \{(x,y) \in \mathbb{R}^2 : y = x^2,\ 0 \le x,y \le 1\}$ + $C_2 = \{(x,y) \in \mathbb{R}^2 : x = 1,\ 1 \le y \le 2\}$



- $\vec{r}_1(x) = \langle x, x^2 \rangle \quad 0 \le x \le 1 \qquad \vec{r}_1\,'(x) = \langle 1, 2x \rangle \qquad |\vec{r}\,'(x)| = \sqrt{1+4x^2}\,dx$

$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{1+4x^2}\,dx$

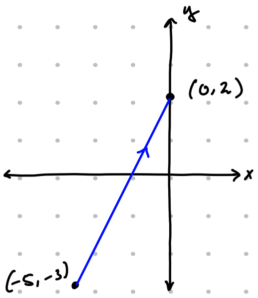
- $\vec{r}_2(y) = \langle 1, y \rangle \quad 0 \le y \le 1 \qquad \vec{r}_2\,'(y) = \langle 0, 1 \rangle \qquad |\vec{r}\,'(y)| = \sqrt{1}\,dy = dy$

$\int_{C_2} 2x\,ds = \int_0^1 2\,dy$

$\int_C 2x\,ds = \int_0^1 2x\sqrt{1+4x^2}\,dx + \int_0^1 2\,dy.$

$\vec{F} \cdot d\vec{r} \qquad \vec{F} = \langle y^2, x \rangle \qquad d\vec{r} = \langle dx, dy \rangle$

Example: Evaluate $\int_C y^2\,dx + x\,dy$ where C is the line segment from $(-5,-3)$ to $(0,2)$.

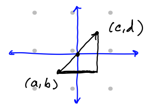


$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$\vec{v} = \langle 5, 5 \rangle$

$P(x_1,y_1) \quad Q(x_2,y_2)$

$\overrightarrow{PQ} = \langle x_2 - x_1,\ y_2 - y_1 \rangle$



$1\vec{i} + 1\vec{j}$

$2\vec{i} + 2\vec{j}$

$\langle c-a,\ d-b \rangle$

$\vec{r}(t) = \langle -5, -3 \rangle + \langle 5t, 5t \rangle$

$x(t) = 5t - 5 \qquad y(t) = 5t - 3 \qquad 0 \le t \le 1$

$dx = 5\,dt$

$\int_0^1 (5t-3)^2\,5\,dt + (5t-5)\,5\,dt = 5\int_0^1 (25t^2 - 25t + 4)\,dt = -\frac{5}{6}$

The Fundamental Theorem of Line Integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

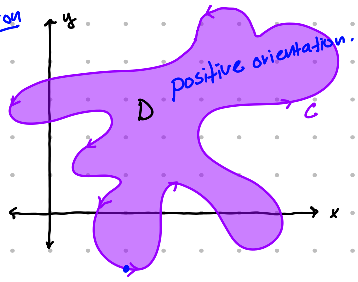
Conservative: $\exists\, f$ such that $\vec{F} = \nabla f$

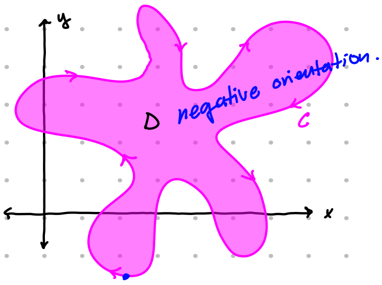
Theorems about path-independence:

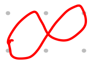
- $\int_C \vec{F} \cdot d\vec{r}$ is path-independent in $D \iff \int_C \vec{F} \cdot d\vec{r} = 0\ \forall$ closed paths in D
- $\int_C \vec{F} \cdot d\vec{r}$ is path-independent in $D \Rightarrow \exists\, f$ such that $\nabla f = \vec{F}$ ($\vec{F}$ is conservative)
- $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is conservative $\iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Section 16.4: Green's Theorem

orientation



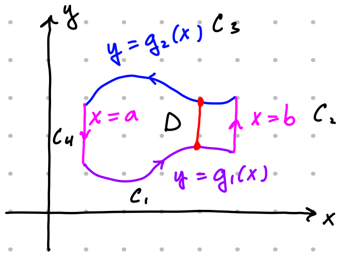




Green's Theorem: Let C be a positively oriented, simple, piecewise-smooth, closed curve in the plane and let D be the region in the plane bounded by C. Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

"Proof"



$$D = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D \frac{\partial P}{\partial y}(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$$

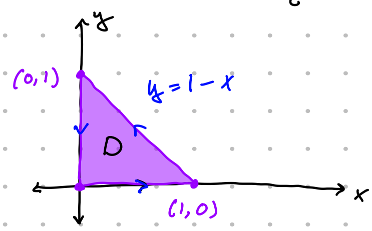
$$= \boxed{\int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx} \quad \text{FTC}$$

Also: $\int_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx + \underbrace{\int_{C_2} P(x,y)\,dx}_{0} + \int_{C_3} P(x,y)\,dx + \underbrace{\int_{C_4} P(x,y)\,dx}_{0}$

$$= \boxed{\int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx}$$

$\Rightarrow \iint_D \frac{\partial P}{\partial y}(x,y)\,dA = \int_C P(x,y)\,dx$ as desired. $\square$

Example: Evaluate $\int_C x^4\,dx + xy\,dy$ where C is the triangle with corners at $(0,0)$, $(1,0)$, and $(0,1)$.



$P(x,y) = x^4 \qquad \frac{\partial P}{\partial y} = 0$

$Q(x,y) = xy \qquad \frac{\partial Q}{\partial x} = y$

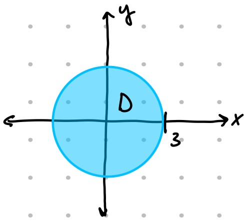
$$\iint_D y\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx$$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\,dx = \tfrac{1}{2}\int_0^1 (x^2 + 1 - 2x)\,dx$$

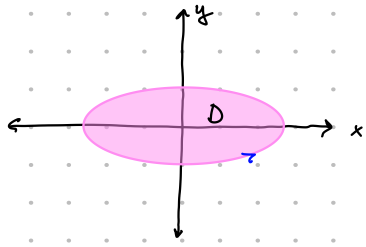
$$= \tfrac{1}{2} \cdot \tfrac{1}{3} + \tfrac{1}{2} - \tfrac{1}{2} = \tfrac{1}{6}$$

Example: Evaluate $\oint_C \underbrace{(3y - e^{\sin(x)})}_{P}\,dx + \underbrace{(7x + \sqrt{y^4+1})}_{Q}\,dy$ where C is the circle $x^2 + y^2 = 9$



$\frac{\partial P}{\partial y} = 3 \qquad \frac{\partial Q}{\partial x} = 7 \qquad \iint_D 4\,dA = 4\iint_D dA = 36\pi$

Example: Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



- Green's Theorem says:

$$\iint_D \Big(\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{}\overset{= 1}{\Big)} dA = \int_C P\,dx + Q\,dy$$

$x(t) = a\cos(t)$

$y(t) = b\sin(t)$

$x^2 = a^2\cos^2 t$

$y^2 = b^2\sin^2(t)$

- So we want $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$. For example, we can let $Q = x$ + $P = 0$.

$$\text{Area} = \int_0^{2\pi} ab\cos^2(t)\,dt \qquad dy = b\cos(t)\,dt$$

$$= \frac{ab}{2}\left(t + \sin(t)\cos(t)\right)\Big|_0^{2\pi} = \frac{ab}{2}(2\pi) = ab\pi$$

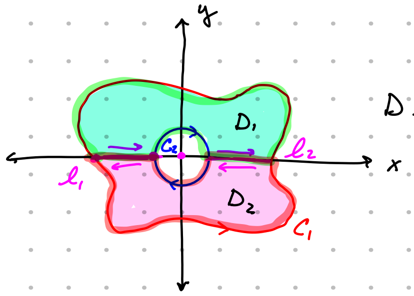
$\vec{F} = \langle P, Q\rangle \qquad \vec{F}\cdot d\vec{r} = P\,dx + Q\,dy$

Example: Show that if $\vec{F}(x,y) = \left\langle \overset{= P}{\frac{-y}{x^2+y^2}}, \overset{= Q}{\frac{x}{x^2+y^2}} \right\rangle$ $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for <u>every</u> positively oriented simple closed curve that encloses the origin.

Hint:

$P(x,y) = \frac{-y}{x^2+y^2} \qquad \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$Q(x,y) = \frac{x}{x^2+y^2} \qquad \frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$



$D = D_1 \cup D_2$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$

$$\iint_{D_1}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0 \qquad \int_{+\ell_1} f(x,y)\,ds = -\int_{-\ell_1} f(x,y)\,ds$$

$$* \qquad 0 = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \boxed{\underbrace{\int_{C_1} P\,dx + Q\,dy}_{\int_{C_1}\vec{F}\cdot d\vec{r}}} - \int_{C_2} P\,dx + Q\,dy$$

unit circle

$-\int_0^{2\pi}$

$\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$$

$P(\cos(t), \sin(t)) = -\sin(t) \qquad dx = -\sin(t)\,dt$

$dy = \cos(t)\,dt$

$Q(\cos(t), \sin(t)) = \cos(t)$

$$\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle \qquad \int_0^{2\pi}\left(\sin^2(t)\,dt + \cos^2(t)\,dt\right) = \int_0^{2\pi} dt = 2\pi$$

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{C_1} P\,dx - \int_{C_2} Q\,dy}$$

Break: X:02

Section 16.5 : Curl + Divergence

Definitions:

$\vec{F} = \langle P, Q, R \rangle$ "vector" $= \langle \partial_x, \partial_y, \partial_z \rangle = \vec{\nabla}$ A vector operator.

$\text{Div}\,\vec{F} = \vec{\nabla} \cdot \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$$\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

*Watch this video!! (Seriously)

https://www.youtube.com/watch?v=rB83DpBJQsE&ab_channel=3Blue1Brown

* And this one (if you like physics)

https://www.youtube.com/watch?v=UzW_jAJzlgI&ab_channel=TheScienceAsylum

*Khan Academy (for even more videos)

Divergence: https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/divergence-grant-videos/v/divergence-intuition-part-1

Curl: https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/curl-grant-videos/v/2d-curl-intuition

- $\vec{\nabla} \cdot \vec{F}$: Is "fluid" created or destroyed?
- $\vec{\nabla} \times \vec{F}$: Rotation.

□ Curl

Example: $\vec{F}(x,y,0) = \overset{P}{xy}\,\vec{i} + \overset{Q}{x}\,\vec{j} + \overset{R}{0}\,\vec{k}$. Find $\vec{\nabla} \times \vec{F}$

$$\begin{vmatrix} \vec{i} & \vec{j} & k \\ \partial_x & \partial_y & \partial_z \\ xy & x & 0 \end{vmatrix} = (0)\vec{i} - (0)\vec{j} + \Big(\underbrace{\frac{\partial Q}{\partial x}}_{1} - \underbrace{\frac{\partial P}{\partial y}}_{x}\Big)\vec{k}$$

Theorem: If f is a function of 3 variables that has continuous 2nd order partial derivatives, then

$$\vec{\nabla} \times \nabla f = \text{curl}(\nabla f) = 0$$

Example: Show that $\vec{F}(x,y,z) = \overset{}{\underset{P}{xz}}\,\vec{i} + \underset{Q}{xyz}\,\vec{j} - \underset{R}{y^2}\,\vec{k}$ is <u>not</u> conservative.

$$\begin{vmatrix} \vec{i} & \vec{j} & k \\ \partial_x & \partial_y & \partial_z \\ xz & xyz & -y^2 \end{vmatrix} = \vec{i}(-2y - xy) - \vec{j}(0 - x) + \vec{k}(yz - 0)$$

$= \langle -2y - xy, x, yz \rangle \neq 0$ $\therefore$ by the theorem, $\vec{F} \neq \nabla f$ for any f.

$\int_C \vec{F} \cdot dr$

Example: Show that $\vec{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ is conservative + find f such that $\nabla f = \vec{F}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \vec{i}(6xyz^2 - 6xyz^2) - \vec{j}(3y^2z^2 - 3y^2z^2) + \vec{k}(2yz^3 - 2yz^3) = 0$$

$\therefore$ $\vec{F}$ is conservative

- $f(x,y,z)$ satisfies $\nabla f = \vec{F}$ $\quad \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$

Then
$$\frac{\partial f}{\partial x} = y^2z^3 \rightsquigarrow xy^2z^3 + C_1(y,z) = f(x,y,z)$$
$$\frac{\partial f}{\partial y} = 2xyz^3 \rightsquigarrow xy^2z^3 + C_2(x,z) = f(x,y,z)$$
$$\frac{\partial f}{\partial z} = 3xy^2z^2 \rightsquigarrow xy^2z^3 + C_3(x,y) = f(x,y,z)$$

constant

$$\boxed{f(x,y,z) = xy^2z^3 + C}$$

"potential function"

□ Divergence $\vec{\nabla}\cdot\vec{F}$

Theorem: If $\vec{F} = \langle P, Q, R\rangle$ + $P, Q, + R$ have cont. 2^{nd} order partials, then

$$\text{div curl}\vec{F} = 0$$
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$$

Example: Show that $\vec{F}(x,y,z) = \langle xz, xyz, -y^2\rangle$ can't be written as the curl of something.

$$\vec{F} \neq \vec{\nabla}\times\vec{G} \text{ for any } \vec{G}.$$

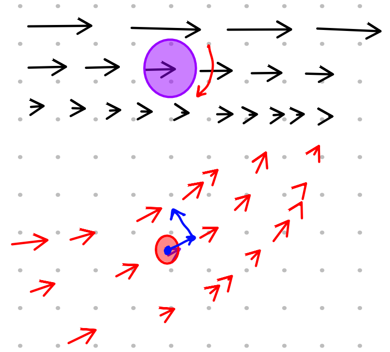
Assume $\vec{F} = \vec{\nabla}\times\vec{G}$. then
$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{G}) = 0$$

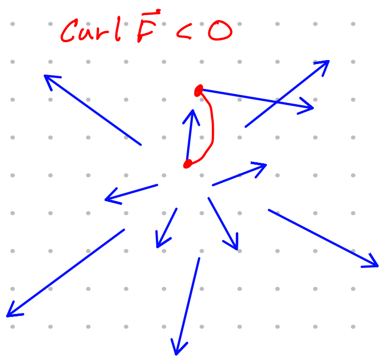
But $\vec{\nabla}\cdot\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle \cdot \langle xz, xyz, -y^2\rangle = \langle z, xz, 0\rangle \neq 0$ □

* Tommorrow: Vector forms of Green's Theorem.

* I will drop the lowest 2 WW grades.

* If you need more time for WW let me know.





Participation Points

1. Let $\vec{F} = \langle x, y, z \rangle$ Find $\vec{\nabla} \times \vec{F}$ and $\vec{\nabla} \cdot \vec{F}$