Thursday, July 21, 2022

MATH 164 Lecture Notes

Review for Midterm 2

Min/Max Problems

~~Problem~~ 1: Find and classify all the critical points of $f(x,y) = (y-2)x^2 - y^2$

Problem 2: Find and classify all the critical points of $f(x,y) = (3x + 4x^3)(y^2 + 2y)$

~~Problem~~ 3: Show the point $(0,0)$ is a critical point of the function $f(x,y) = x^4 + 6y^2 - 4xy^3 - 1$ and prove that it is a local minimum.

~~Problem~~ 4: Find the absolute max/min $f(x,y) = (9x^2-1)(1-4y)$ on the rectangle $[-2,3] \times [-1,4]$

Problem 5: Find the min/max values of $f(x,y) = 81x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$

~~Problem~~ 6: The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to + farthest from the origin.

~~Problem~~ 7: Find + classify the critical points of $f(x,y) = x^2 + 4y^2 + 4xy + 2$

~~Problem~~ 8: Find the critical points of $f(x,y) = x^2 + x - \sin(xy)$

Problem 1: Find and classify all the critical points of $f(x,y) = (y-2)x^2 - y^2$

$f_x(x,y) = 2x(y-2) = 0 \Rightarrow 2x = 0$ or $y = 2 \rightsquigarrow (0,0)$

$f_y(x,y) = x^2 - 2y = 0$ when $y = 2$, $x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \rightsquigarrow (2,2)\ (-2,2)$

Critical points: $\{(0,0), (2,2), (-2,2)\}$

Classify: Use 2nd derivative test

$f_{xx} = 2(y-2) = 2y - 4$ $\quad f_{yy} = -2$
$f_{xy} = 2x$

$$D(x,y) = \begin{vmatrix} 2y-4 & 2x \\ 2x & -2 \end{vmatrix} = -4y + 8 - 4x^2$$

$D(0,0) = 8 > 0$
$f_{xx}(0,0) = -4 < 0$ $\Rightarrow$ $(0,0)$ is a local max

$D(2,2) = -8 + 8 - 4 \cdot 4 = -16 < 0 \rightsquigarrow (2,2)$ is a saddle point

$D(-2,2) = -8 + 8 - 4 \cdot 4 = -16 < 0 \rightsquigarrow (2,2)$ is a saddle point

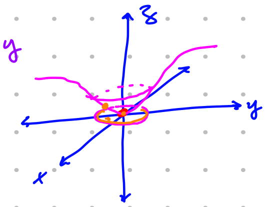
Problem 3: Show the point $(0,0)$ is a critical point of the function $f(x,y) = x^4 + 6y^2 - 4xy^3 - 1$ and prove that it is a local minimum.

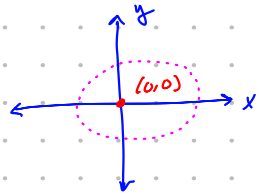
- Show the point $(0,0)$ is a critical point

$f_x(x,y) = 4x^3 - 4y^3$

$f_y(x,y) = 12y - 12xy^2$

Clearly $(0,0)$ satisfies $f_x(0,0) = f_y(0,0) = 0$





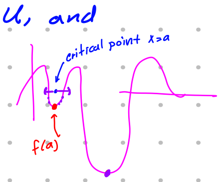
2^{nd} derivative test? $f_{xx} = 12x^2$ $f_{xy} = -12y^2$ $f_{yy} = 12 - 24xy$

$$D(0,0) = \begin{vmatrix} 0 & 0 \\ 0 & 12 \end{vmatrix} = 0$$

$f(x,x) = x^4 + 6x^2 - 4x^4 - 1 = -3x^4 + 6x^2 - 1$

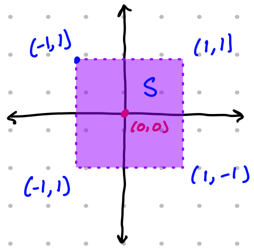
small disk, or square or whatever.

* $(0,0)$ is a local min of $f(x,y)$ if there exist a neighborhood $U \subset \mathbb{R}^2$ such that $(0,0) \in U$, and $f(x,y) \geq f(0,0)$ for all $(x,y) \in U$.



* $f(0,0) = -1$

$f(x,y) = x^4 + 6y^2 - 4xy^3 - 1$



Claim: $f(x,y) \geq -1$ for all $(x,y) \in S = \{(x,y) : |x| \leq 1, |y| \leq 1\}$

$x^4 + 6y^2 - 4xy^3 - 1 \geq -1$ on S

$\Leftrightarrow x^4 + 6y^2 - 4xy^3 \geq 0$

Observation $x^4 + 6y^2 - 4xy^3 \geq 6y^2 - 4xy^3$ since $x^4 \geq 0$

$\therefore$ It suffices to show that $6y^2 - 4xy^3 \geq 0$ for all $(x,y) \in S$.

$6y^2 - 4xy^3 = 2y^2(3 - 2xy) \geq 0$

$\Leftrightarrow 3 - 2xy \geq 0 \Leftrightarrow \frac{3}{2} \geq xy$ But if $|x| \leq 1$ & $|y| \leq 1$, then $|xy| = |x||y| \leq 1$

$\therefore xy \leq \frac{3}{2}$ done. $\square$

Problem 7: Find + classify the critical points of $f(x,y) = x^2 + 4y^2 - 4xy + 2$

$$\left.\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= 8y - 4x = 0 \rightsquigarrow 4y - 2x = 0 \end{aligned}\right\}$$

$2y = x = 0$ Critical points: $\{(x,y) : x = 2y\}$

$y = \frac{1}{2}x$

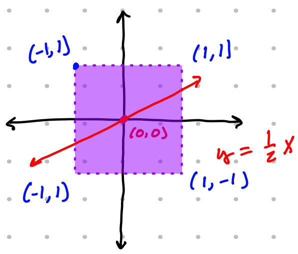
$f_{xx} = 2$ $f_{xy} = -4$

$f_{yy} = 8$

$$\begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix} = 16 - 16 = 0$$ 2^{nd} derivative test fails.

$f(x,y) = x^2 + 4y^2 - 4xy + 2 = (x - 2y)^2 + 2$

When $x = 2y$, $\underbrace{4y^2 + 4y^2}_{8y^2} - \underbrace{4 \cdot 2y^2}_{-8y^2} + 2 = 2$



$f(1,0) = 1^2 + 2 = 3 > 2 \rightsquigarrow$ Conjecture that this a minimum.

$f(x,y) = (x - 2y)^2 + 2 \geq 2$ $\forall$ values of x.

$\therefore$ Critical points: $\{(x,y) : x = 2y\}$ all Global minimums.

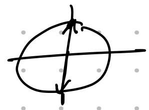
Problem 8: Find the critical points of $f(x,y) = x^2 + x - \sin(xy)$

$f_x(x,y) = 2x + 1 - \cancel{y\cos(xy)} = 0$

$f_y(x,y) = -x\cos(xy) = 0 \rightsquigarrow x = 0$ or $\cos(xy) = 0$

when $x=0$, $1 - y = 0 \Rightarrow y = 1$

when $\cos(xy) = 0$ $\quad 2x+1=0 \Rightarrow x = -1/2$



$\cos(\frac{-y}{2}) = 0 \Rightarrow \frac{-y}{2} = \frac{\pi}{2} + 2\pi n$ or $\frac{3\pi}{2} + 2\pi n$

$y = -\pi - 4\pi n$ or $-3\pi - 4\pi n$

$y = -\pi - 4\pi n$ or $-3\pi - 4\pi n$

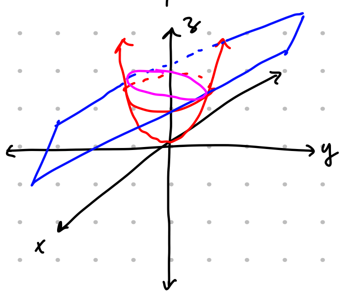
CP: $(0,1)$

$(-1/2, -\pi - 4\pi n)$

$(-1/2, -3\pi - 4\pi n)$

$\forall n \in \mathbb{Z}$.

Problem 6: The plane $x+y+2z=2$ intersects the paraboloid $z = x^2+y^2$ in an ellipse. Find the points on this ellipse that are nearest to & farthest from the origin.



* Lagrange multipliers problem.

$f(x,y,z) = x^2 + y^2 + z^2$ where $P(x,y,z)$ is an arbitrary point

$\nabla f = \langle 2x, 2y, 2z \rangle$

$g(x,y,z) = x + y + 2z$ $\qquad x+y+2z = 2$

$\nabla g = \langle 1, 1, 2 \rangle$ $\qquad z = x^2 + y^2$

LMT: $\nabla f = \lambda \nabla g + \mu \nabla h$

$h(x,y,z) = z - x^2 - y^2$

$\nabla h = \langle -2x, -2y, 1 \rangle$

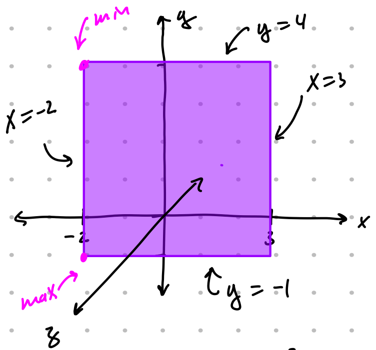
$\left.\begin{array}{l} 2x = \lambda - 2\mu x \\ 2y = \lambda - 2\mu y \end{array}\right\}$ $\quad 2x + 2\mu x = 2y + 2\mu y \qquad x(2+2\mu) = y(2+2\mu) \Rightarrow \mu = -1$ or $x = y$

$2z = 2\lambda + \mu$

$x + y + 2z = 2$

$z = x^2 + y^2$

Problem 4: Find the absolute max/min $f(x,y) = (9x^2-1)(1-4y)$ on the rectangle $[-2,3] \times [-1,4]$



$f_x = 18x(1-4y) = 0 \Rightarrow \cancel{x=0}$ or $y = 1/4$

$f_y = -4(9x^2-1) = -36x^2 + 4 = 0 = -9x^2 + 1 = 0$

If $x = 0$, contradiction. $\qquad 1 = 9x^2 \Rightarrow x^2 = \frac{1}{9} \Rightarrow x = \frac{\pm 1}{3}$

$(\frac{1}{3}, \frac{1}{4})$ & $(-\frac{1}{3}, \frac{1}{4})$

when $x = -2$ $\quad \overset{"z"}{f(-2,y)} = (9 \cdot 4 - 1)(1 - 4y) = 35 - 35 \cdot 4y$

$z = -35 \cdot 4y + 35$ on $[-1, 4]$ $\qquad$ max on this side is $35(5)$

min on this side is $35(-15)$

Break: 06

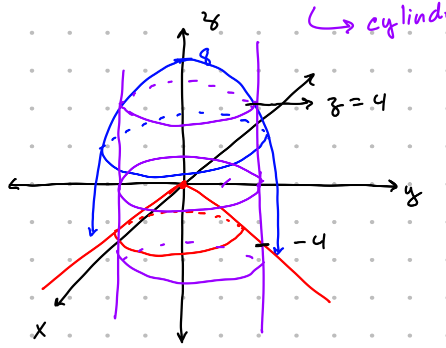
* Multiple Integrals.

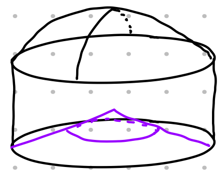
Problem 9: Use a triple integral to find the volume below $z = 8 - x^2 - y^2$, above $z = -\sqrt{4x^2 + 4y^2}$ and inside $x^2 + y^2 = 4$.

paraboloid facing down

cone facing down

$\hookrightarrow$ cylinder.

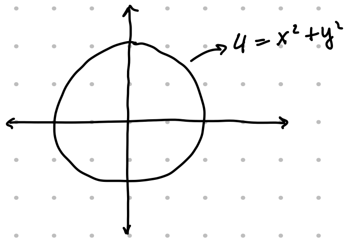




paraboloid intersect cylinder:

$z = 4$

$z = -\sqrt{4 \cdot 4} = -4$



$$\int_0^{2\pi} \int_0^2 \int_{-2r}^{8-r^2} r \, dz \, dr \, d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= 2\pi \int_0^2 \int_{-2r}^{8-r^2} r \, dz \, dr$$

$$= 2\pi \int_0^2 \left(r(8-r^2) - r(-2r) \right) dr$$

$$= 2\pi \int_0^2 (8r - r^3 + 2r^2) \, dr$$

$$= 2\pi \left(16 - \frac{2^4}{4} + \frac{2}{3} \cdot 2^3 \right)$$

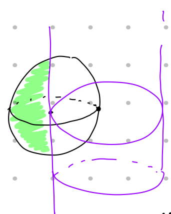
Problem ~~7~~ 10

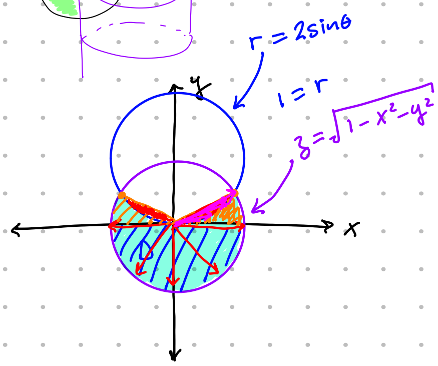
Set up but **do not evaluate** the integral representing the volume of the solid contained **inside** the sphere $1 = x^2 + y^2 + z^2$ and **outside** the cylinder $x^2 + (y-1)^2 = 1$

$y^2 - 2y + 1$

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + \cancel{1} = \cancel{1}$

cylinder w/ center: $(0, 1)$ + radius 1





$$2 \iint_D \sqrt{1 - x^2 - y^2}$$

$x = r\cos\theta$

$y = r\sin\theta$

① $2 \int_\pi^{2\pi} \int_0^1 \sqrt{1 - r^2} \cdot r \, dr \, d\theta$

② $4 \int_0^{\pi/6} \int_0^1 \sqrt{1 - r^2} \cdot r \, dr \, d\theta$

③ $4 \int_0^{\pi/6} \int_0^{2\sin\theta} \sqrt{1 - r^2} \cdot r \, dr \, d\theta$ (subtract)

Circles intersect when $2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

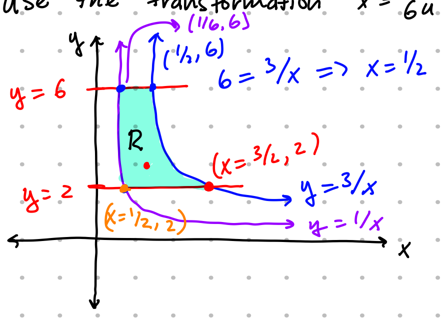
$\Rightarrow \theta = \pi/6$ or $5\pi/6$

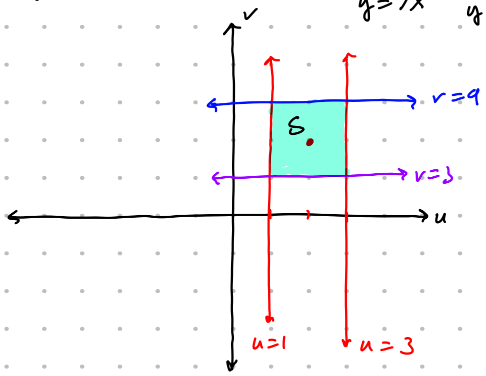
$$2\int_{\pi}^{2\pi}\int_0^1 \sqrt{1-r^2}\cdot r\,dr\,d\theta + 4\int_0^{\pi/6}\int_0^1 \sqrt{1-r^2}\cdot r\,dr\,d\theta - 4\int_0^{\pi/6}\int_0^{2\sin\theta}\sqrt{1-r^2}\cdot r\,dr\,d\theta$$

Problem 11: Evaluate $\iint_R xy^3\,dA$ where R is bounded by $xy=1$, $xy=3$, $y=2$, $y=6$

$xy=1 \Rightarrow y = 1/x$, $xy=3 \Rightarrow y=3/x$

use the transformation $x = \frac{v}{6u}$, $y = 2u$.





$x = \frac{v}{6u}$, $y = 2u$.

$\Rightarrow \boxed{u = \frac{y}{2}}$

$x = \frac{v}{6}\cdot\frac{2}{y} = \frac{v}{3y} \Rightarrow \boxed{3yx = v}$

when $y=6$, $u=3$

when $y=2$, $u=1$

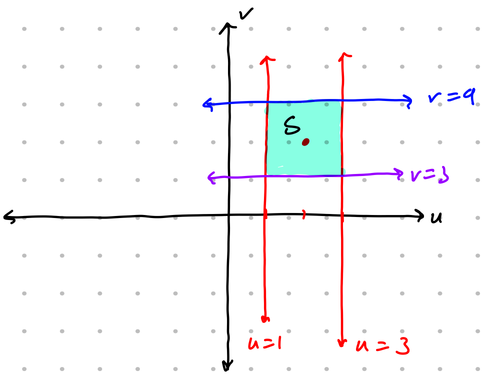
when $xy=1$, $v=3$

when $xy=3$, $v=9$

$x = \frac{v}{6u}$, $y=2u$.

$$\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} -v/6u^2 & 1/6u \\ 2 & 0 \end{vmatrix} = -\frac{1}{3u}$$

$$\iint_R xy^3\,dA = \int_3^9\int_1^3 \frac{v}{6u}\cdot 8u^3\,\frac{1}{3u}\cdot du\,dv = \boxed{\int_3^9\int_1^3 \frac{4}{9}\,vu\,du\,dv}$$



$$= \frac{4}{9}\int_3^9 \frac{1}{2}v\cdot u^2\Big|_1^3\,dv$$

$$= \frac{2}{9}\int_3^9 8v\,dv = \frac{16}{9}\cdot\frac{1}{2}(81-9)$$