Wednesday, July 20, 2022

MATH 164 Lecture Notes

Chapter 16: Vector Calculus * Not on Midterm 2.

Section 16.1: Vector Fields

- **Definition:** Let $D \subseteq \mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is function $\vec{F}$ that assigns a vector $\vec{F}(x,y)$ to each point $(x,y) \in \mathbb{R}^2$.

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = \langle P(x,y), Q(x,y) \rangle$$

- **Definition:** Let $D \subseteq \mathbb{R}^n$. A vector field in $\mathbb{R}^n$ is function $\vec{F}$ that assigns a vector $\vec{F}(\vec{x})$ to each point $\vec{x} \in \mathbb{R}^n$

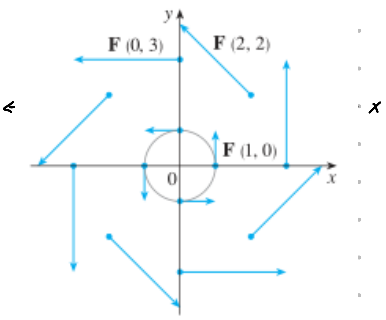
$\vec{x} = (x_1, \dots, x_n)$

$$\vec{F}(\vec{x}) = \langle P_1(\vec{x}), \dots, P_n(\vec{x}) \rangle$$

function of n variables

Example: $\vec{F}(x,y) = \langle -y, x \rangle$

At $(1,0)$, $\vec{F}(1,0) = \langle 0, 1 \rangle$
$\vec{F}(2,0) = \langle 0, 2 \rangle$
$\vec{F}(1,1) = \langle -1, 1 \rangle$



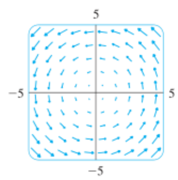


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$

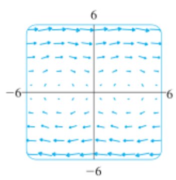


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

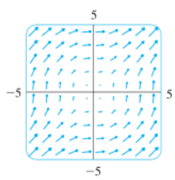


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

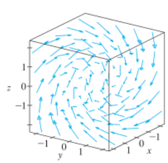


FIGURE 10
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

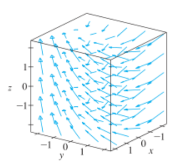


FIGURE 11
$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

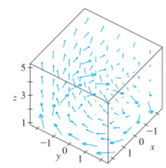


FIGURE 12
$\mathbf{F}(x, y, z) = \frac{y}{z}\,\mathbf{i} - \frac{x}{z}\,\mathbf{j} + \frac{z}{4}\,\mathbf{k}$

- Gradient vector fields of functions (conservative)

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

A vector field $\vec{F}$ is called conservative if $\exists$ a function f such that $\vec{F} = \nabla f$. In this case, f is called the potential function for $\vec{F}$.

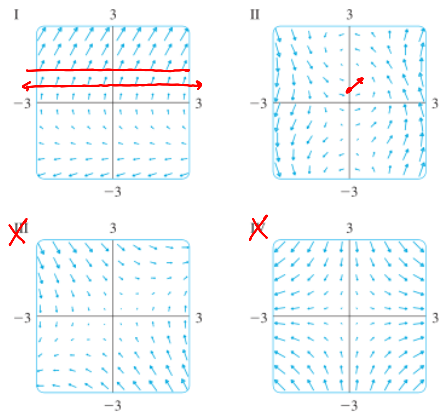
11–14 Match the vector fields **F** with the plots labeled I–IV. Give reasons for your choices.

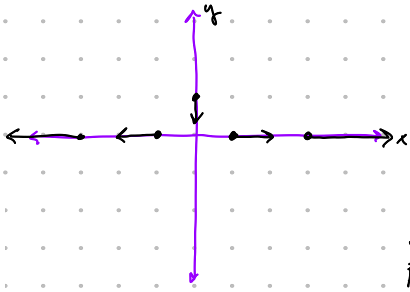
11. $\mathbf{F}(x, y) = \langle x, -y \rangle$ $\vec{F}(x,y) = \langle x, -y \rangle$ IV

12. $\mathbf{F}(x, y) = \langle y, x - y \rangle$ III

13. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$ $F(0,1) = \langle 1, 3 \rangle$ I

14. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$ II





$\vec{i} = \langle 1, 0 \rangle$

$\vec{j} = \langle 0, 1 \rangle$

$\vec{F}(1,0) = \vec{i}$

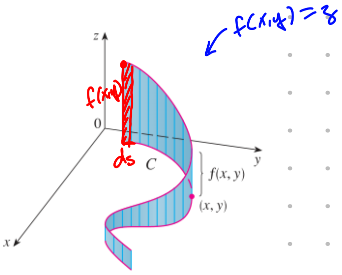
$\vec{F}(a,0) = \langle a, 0 \rangle$

$F(-1,0) = \langle -1, 0 \rangle$

$F(0,1) = \langle 0, -1 \rangle$

$F(0,-1) = \langle 0, 1 \rangle$

Section 16.2: Line Integrals



Definition: If f is defined on a smooth curve C given by the parametric equations $x(t)$, $y(t)$, then the line integral of f along C is:

$$\int_C f(x,y)\,\underset{\text{arc length}}{ds} = \int_a^b f(x(t),y(t))\underbrace{\sqrt{(dx/dt)^2 + (dy/dt)^2}}_{\text{"Jacobian"}}\,\underset{\text{time}}{dt}$$

t times

(line integral with respect to arc length)

We also have line integrals with respect to x and y:

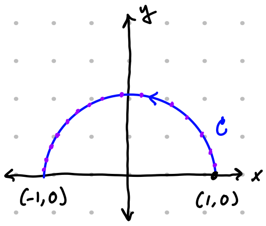
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

C = trace of $\vec{r}(x(t),y(t))$

* In higher dimensions: $\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$ for C given by $\vec{r}(t) = \langle x_1(t), \dots, x_n(t)\rangle$

Example: Evaluate $\int_C (2 + x^2y)\,ds$ where C is the upper half of the unit circle.



$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

1. Parameterize C: $x = \cos(t)$ $\quad y = \sin(t)$ $\quad 0 \le t \le \pi$

2. Find $f(\cos(t), \sin(t))$

$$f(\cos(t), \sin(t)) = 2 + \cos^2(t)\sin(t)$$

3. Find $\sqrt{(dx/dt)^2 + (dy/dt)^2}$

$$\frac{dx}{dt} = -\sin(t) \qquad \frac{dy}{dt} = \cos(t) \qquad \sqrt{(dx/dt)^2 + (dy/dt)^2} = 1$$

4. Set up integral + integrate

$$\int_0^\pi \big(2 + \overbrace{\cos^2(t)\sin(t)}^{-u^2\,du}\big)\,dt \qquad u = \cos(t), \quad du = -\sin(t)$$

$$= 2\pi - \frac{\cos^3 t}{3}\Big|_0^\pi = 2\pi - \frac{2}{3}$$

▫ Line Integrals of Vector Fields

all the component functions are continuous.

Definition: Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by $\vec{r}(t)$, $a \le t \le b$. Then the integral of $\vec{F}$ along C is

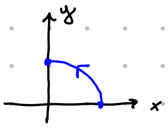
arclength

$$\int_C \vec{F}\cdot d\vec{r} = \boxed{\int_a^b \vec{F}(\vec{r}(t))\cdot \underline{\vec{r}\,'(t)\,dt}} = \int_C \vec{F}\cdot \vec{T}\,ds$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz \quad \text{where} \quad \vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}}$$

Calculate $\int_C \vec{F}\cdot d\vec{r}$

Example: Find the work done by the force field $\vec{F}(x,y) = \langle \underline{x}^2, \underline{-xy} \rangle$ in moving a particle along the quarter circle $\vec{r}(t) = \langle \underline{\cos(t)}, \underline{\sin(t)} \rangle$ $0 \le t \le \pi/2$



$$\vec{F}(\vec{r}(t)) = \langle \cos^2(t), -\cos(t)\sin(t) \rangle$$
$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$$

$$\vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t) = -\cos^2(t)\sin(t) - \cos^2(t)\sin(t) = -2\cos^2(t)\sin(t)$$

$$\int_0^{\pi/2} -2\cos^2(t)\sin(t)\,dt = \frac{2\cos^3(t)}{3}\Bigg|_0^{\pi/2} = -\frac{2}{3}$$

Section 16.3: The Fundamental Theorem for Line Integrals

* Recall the FTC: $\int_a^b F'(x)dx = F(b) - F(a)$

Theorem: Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differential function of two or three variables whose gradient vector field ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof:

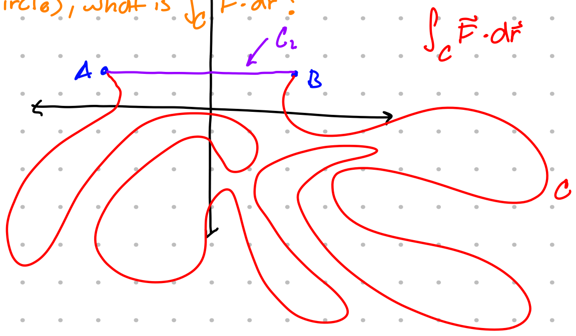
$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$ regular FTC

* If $\vec{F}$ is conservative ($\vec{F} = \nabla f$), + C is a closed curve, (unit circle), what is $\int_C \vec{F} \cdot d\vec{r}$?



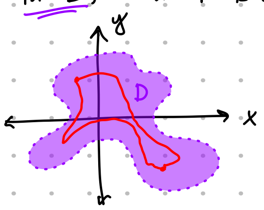
□ Path Independence

$$\int_{C_1} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = \int_{C_2} \nabla f \cdot d\vec{r}$$ (when C_1 + C_2 both start/end at the same point + ∇f is continuous on C_1 + C_2)

* If a vector field is conservative, it's also path independent.

Theorem: $\int_C \vec{F} \cdot d\vec{r}$ is independent of path (conservative) if + only if $\int_C \vec{F} \cdot d\vec{r} = 0$ $\forall$ closed paths C.

Theorem: Suppose $\vec{F}$ is a vector field that is continuous on an open, connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is path-independent in D, then $\vec{F}$ is conservative on D. That is, $\exists$ f such that $\vec{F} = \nabla f$.



$\exists$ f such that $\vec{F} = \nabla f$ $\iff$ path independent

$\iff$ $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ $\quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Example: Determine whether $\vec{F}$ is conservative. Find f such that $\nabla f = \vec{F}$

$$\vec{F}(x,y) = \langle \underbrace{3 + 2xy}_{P(x,y)}, \underbrace{x^2 - 3y^2}_{Q(x,y)} \rangle$$

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x} = 2x \implies \vec{F} \text{ is conservative.}$$

• If $\nabla f = \vec{F}$, then

Want to find f such that $\nabla f = \langle f_x(x,y), f_y(x,y) \rangle = \vec{F}$

regular function

$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = 3 + 2xy$ ⤳ pretend y is constant + integrate wrt x:

* $f(x,y) = \int 3 + 2xy\,dx = 3x + x^2y + C(y)$

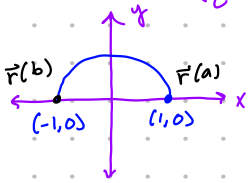
$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = x^2 - 3y^2$ ⤳ pretend x is constant + integrate wrt y:

* $f(x,y) = \int (x^2 - 3y^2)dy = x^2y - y^3 + K(x)$

$$\implies f(x,y) = \boxed{3x} + x^2y + \boxed{C(y)} = x^2y \boxed{-y^3} + \boxed{K(x)} \implies \boxed{f(x,y) = x^2y + 3x - y^3 + K}$$

$f_x = 2xy + 3$

Now Find $\int_C \vec{F} \cdot d\vec{r}$ where C is the upper ½ unit circle.



$$\int_C \overset{\vec{F}}{\nabla f} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$f(\vec{r}(b)) = f(-1,0) = -3 + K$$

$$f(\vec{r}(a)) = f(1,0) = 3 + K$$

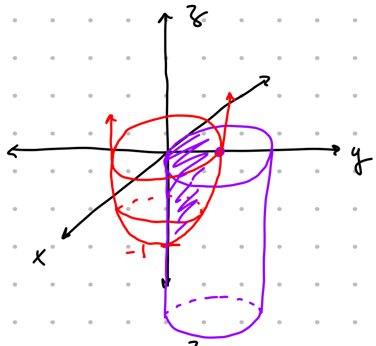
$$\int_C \nabla f \cdot d\vec{r} = -3 + \cancel{K} - 3 - \cancel{K} = -6$$

* Break: X:15

Problem 3

Part a (7 points): Set up an integral whose value is equal to the volume of the solid that is *inside* the cylinder $x^2 + y^2 = 2y$, *below* the plane $z = 0$ and *above* the surface $z = x^2 + y^2 - 1$.

Part b (3 points): Reduce the integral to the integral of a function of a single variable. You do *not* have to calculate the 1-dimensional integral.

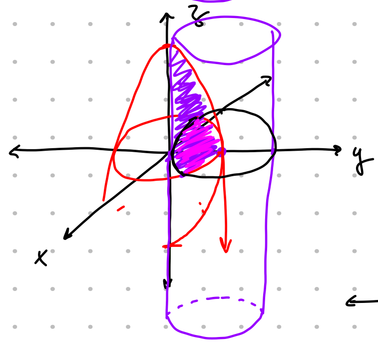


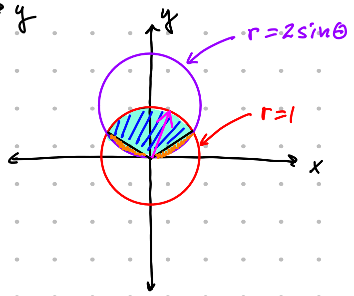
Cylinder: $r^2 = 2r\sin(\theta)$

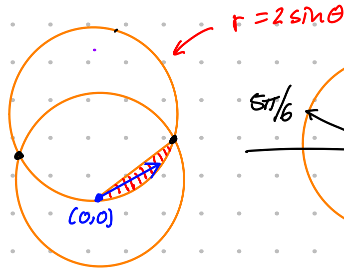
$r = 2\sin\theta$

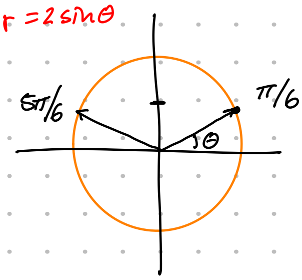
paraboloid: $z = x^2+y^2-1 \qquad z = r^2 - 1$

$$\boxed{\int_{\pi/6}^{5\pi/6}\int_0^1 (r^2-1)\, r\, dr\, d\theta + 2\int_0^{\pi/6}\int_0^{2\sin\theta} (r^2-1)\, r\, dr\, d\theta}$$









$1 = 2\sin\theta \Rightarrow \sin\theta = \frac{1}{2}$

Problem 4: Evaluate the integrals

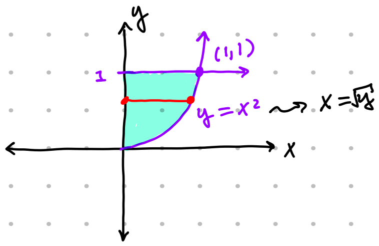
(You may assume that Fubini's theorem applies) ← Hint to switch order of integration.

Part a (5 points):

$$\int_0^1 \int_{x^2}^1 \sqrt{y}\sin(y)\,dy\,dx$$

Part b (5 points):

$$\int_0^1 \int_y^1 x^{-3/2}\cos\left(\frac{\pi y}{2x}\right)dx\,dy$$



$$\int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\sin(y)\,dx\,dy$$

$$= \int_0^1 \sqrt{y}\sin(y)\cdot x\Big|_0^{\sqrt{y}}\,dy$$

$$= \int_0^1 y\sin(y)\,dy$$

D	I
y +	$\sin y$
1 −	$-\cos y$
0	$-\sin(y)$

$$= -y\cos(y) + \sin(y)\Big|_0^1$$

$$= -\cos(1) + \sin(1)$$

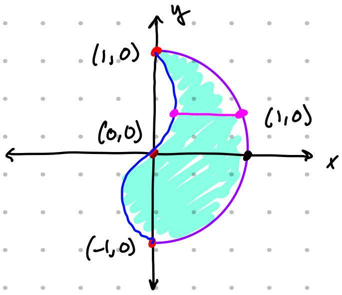
Problem 6

$y = x - x^3$

Evaluate the integral $\int\int_D y dA$ where D is the region in R^2 bounded by $x = y - y^3$ and $x = \sqrt{1 - y^2}$

$x^2 = 1 - y^2$

$x^2 + y^2 = 1$



$y - y^3 = 1 - y^2$

$y(\cancel{1-y^2}) = \cancel{1-y^2}$

$y = \pm 1$

$$\int_{-1}^{1} \int_{y-y^3}^{\sqrt{1-y^2}} y \, dx \, dy$$

Problem 7

Set up but **do not evaluate** the integral representing the volume of the solid contained **inside** the sphere $1 = x^2 + y^2 + z^2$ and **outside** the cylinder $x^2 + (y-1)^2 = 1$