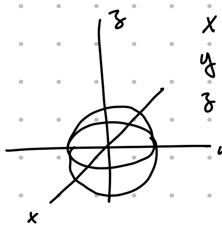
Tuesday, July 19, 2022
MATH 164 Lecture Notes

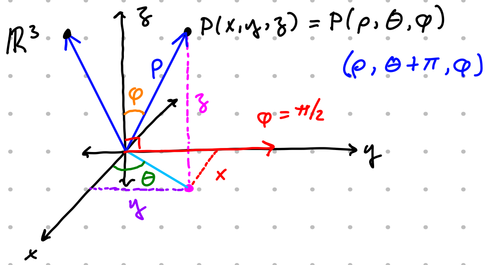


$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

Section 15.8: Triple Integrals in Spherical Coordinates

remember the force, reserve the fuel,

★ Conventions:



- "ρ" is the Greek letter "rho", for "radius"
- "θ" is the Greek letter "theta" and represents the angle from the x-axis
- "φ" is the Greek letter "phi" and represents the angle from the z-axis

★ Remember: $\boxed{\rho \geq 0, \quad 0 \leq \varphi \leq \pi, \text{ and } 0 \leq \theta \leq 2\pi}$!

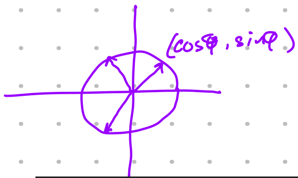
when $\varphi = \pi/2$, we should have polar coordinates. $x = \rho\cos\theta \quad y = \rho\sin\theta \quad z = 0$

$$\boxed{x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi}$$

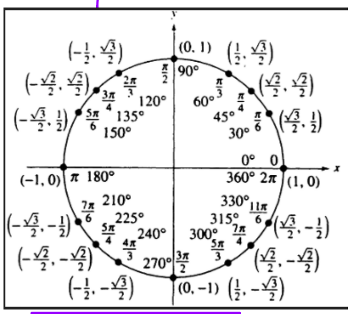
Observation: $x^2 + y^2 + z^2 = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\varphi$

$= \rho^2\sin^2\varphi\,(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\varphi$

$= \rho^2(\sin^2\varphi + \cos^2\theta) = \rho^2$



Example: Convert $P(0, 2\sqrt{3}, -2)$ into spherical coordinates.



$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$

$\rho\sin\varphi\cos\theta = 0 \qquad\qquad \rho^2 = 12 + 4 = 16 \implies \rho = 4$

$\rho\sin\varphi\sin\theta = 2\sqrt{3}$

$\rho\cos\varphi = -2 \rightsquigarrow \cos\varphi = \frac{-2}{4} = -\frac{1}{2} \implies \varphi = \frac{2\pi}{3}$

$\rho\sin\varphi\cos\theta = 0 \implies \theta = \pi/2 \quad \frac{3\pi}{2}$

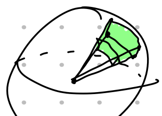
$4\underbrace{\sin(2\pi/3)}_{\text{positive}}\cdot\sin\theta = 2\sqrt{3} \implies \sin\theta$ is positive $\implies \theta = \pi/2$

$$\boxed{(4, \pi/2, 2\pi/3)}$$

Jacobian T: Euclidean $\to$ Spherical

$$\boxed{\text{★Spherical volume element: } dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi}$$

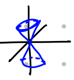
$\iiint_E f(x,y,z)\,dV \qquad dV = r\,dr\,d\theta$



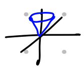
Example: Use spherical coordinates to find the volume above the cone $z = \sqrt{x^2+y^2}$ + below the sphere $x^2 + y^2 + z^2 = z$.

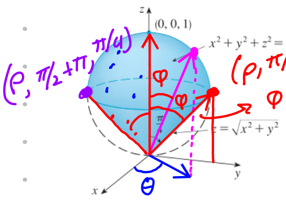
$0 \leq \varphi \leq \pi$

$z^2 = x^2 + y^2 \rightsquigarrow$



$z = \sqrt{x^2+y^2} \rightsquigarrow$





$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$x^2 + y^2 + z^2 = z. \implies \rho^2 = \rho\cos\varphi \implies \boxed{\rho = \cos\varphi}$

$z = \sqrt{x^2+y^2} \rightsquigarrow z^2 = x^2 + y^2 \qquad \rho^2\cos^2\varphi = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta$

$= \rho^2\sin^2\varphi\,(\cos^2\theta + \sin^2\theta) = \rho^2\sin^2\varphi$

$\boxed{\cos\varphi = \sin\varphi} \implies \varphi = \pi/4$

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\varphi} \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi = 2\pi\int_0^{\pi/4}\int_0^{\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi$$

$$= 2\pi\int_0^{\pi/4} \sin\varphi\, \frac{1}{3}\rho^3\Big|_0^{\cos\varphi} d\varphi = \boxed{\frac{2\pi}{3}\int_0^{\pi/4} \sin\varphi\cos^3\varphi\, d\varphi}$$

$$\frac{-\cos^4\theta}{4}$$

$u = \cos\varphi$

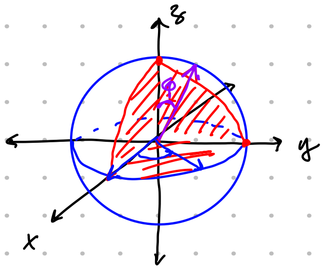
$-du = \sin\varphi\, d\varphi$

$$\frac{-2\pi}{3}\int_A^B u^3\, du = \frac{-\cancel{2}\pi}{3}\cdot\frac{1}{\cancel{4}_2}u^4\Big|_A^B$$

$$\frac{-\pi}{6}\left(\cos^4(\pi/4) - \cos^4(0)\right)$$

$$\left(\frac{\sqrt{2}}{2}\right)^4 \quad - \quad 1$$

25) Evaluate : $\iiint_E x\cdot e^{x^2+y^2+z^2}\,dV$ E is the portion of the unit ball $x^2+y^2+z^2 \le 1$ that lies in the first octant



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3 \sin^2\varphi\cos\theta\, e^{\rho^2}\, d\rho\, d\theta\, d\varphi$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin^2\varphi\cos\theta\, d\theta\, d\varphi \int_0^1 \rho^3 e^{\rho^2} d\rho$$

$$\int w\,dv = wv - \int v\,dw$$

$u = \rho^3$ $\quad du = 3\rho^2 d\rho$

$u = \rho^2$ $\quad du = 2\rho\, d\rho$

$\frac{1}{2}du = \rho\, d\rho$

$$\frac{1}{2}\int_0^1 u e^u\, du$$

$w = u$ $\quad dw = du$

$dv = e^u \Rightarrow v = e^u$

$$\int u e^u = u e^u - e^u$$

double-angle

straightforward.

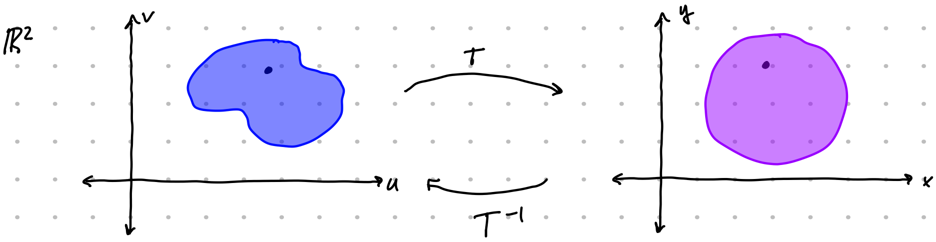
$$\int_0^{\pi/2}\int_0^{\pi/2} \sin^2\varphi\cos\theta\, d\theta\, d\varphi \int_0^1 \rho^3 e^{\rho^2} d\rho = \int_0^{\pi/2} \sin^2\varphi\, d\varphi \cdot \int_0^{\pi/2} \cos\theta\, d\theta \cdot \int_0^1 \rho^3 e^{\rho^2}\, d\rho$$

X : 53

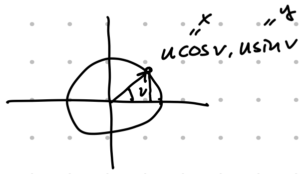
Section 15.9: Change of Variables in Multiple Integrals

* 2-dimensions (mostly)

C^1- transformations:
- 1-1
- invertible
- components have to be differentiable in all variables.



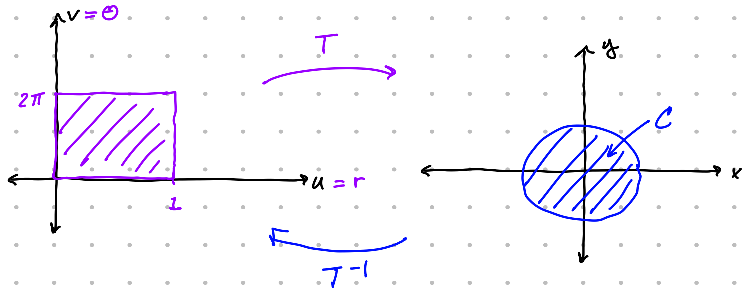
$T(u,v) = (T_1(u,v) = y,\ T_2(u,v) = x)$



$\boxed{\begin{aligned} x &= u\cos v \\ y &= u\sin v \end{aligned}}$ $\rightsquigarrow$ $\sqrt{x^2+y^2} = u$

$v = \tan^{-1}(y/x)$

$T(u,v) = (u\cos v, u\sin v)$



$$\iint_C f(x,y)\,dA$$

Example: A transformation is defined by the equations

Not linear $\longrightarrow$ $x = u^2 - v^2$, $y = 2uv$

$x = -a^2$, $y \geq 0$

$\left.\begin{aligned} x &= a^2 - 1 \\ y &= 2a \rightsquigarrow \tfrac{y}{2} = a \end{aligned}\right\}$ $x = \frac{y^2}{4} - 1$

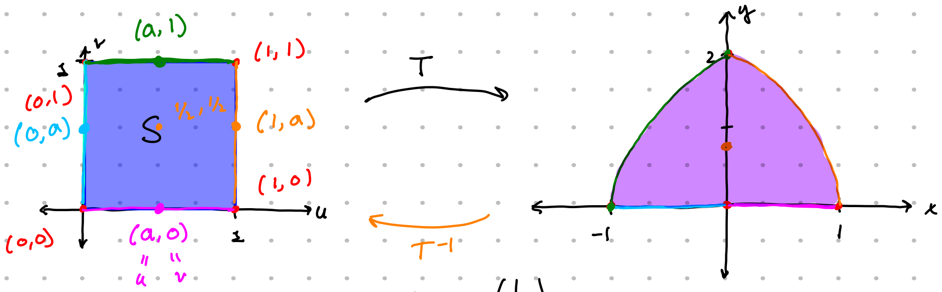
Find the image of the square $S = \{(u,v) : 0 \leq u \leq 1,\ 0 \leq v \leq 1\}$

$1 = \frac{y^2}{4} \Rightarrow y^2 = 4$

$x = 1 - a^2$

$y = 2a$

$x = 1 - \frac{y^2}{4}$

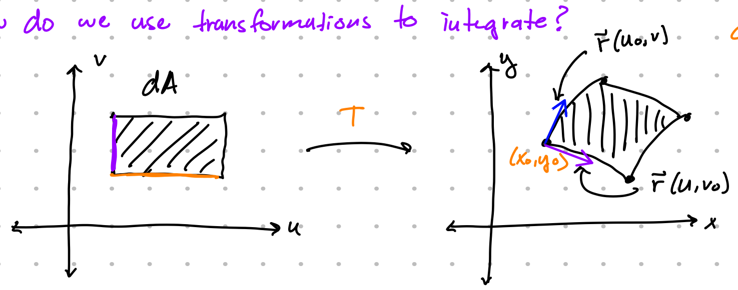


$x = a^2$ $\left(\frac{1}{4^2}\right)$

$y = 0$

* How do we use transformations to integrate?

dA



$\vec{r}(u,v) = g(u,v)\vec{i} + h(u,v)\vec{j}$

$\vec{r}_u(u_0,v_0) = g_u(u_0,v_0)\vec{i} + h_u(u_0,v_0)$

$= \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}$

$\vec{r}_u \times \vec{r}_v$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x/\partial u & \partial y/\partial u & 0 \\ \partial x/\partial v & \partial y/\partial v & 0 \end{vmatrix} = \boxed{\begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}}\vec{k}$$

Jacobian.

Definition: The Jacobian of the transformation T given by $x = g(u,v)$, $y = h(u,v)$ is

Just a number.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}$$

Polar Coordinates:

$x = u\cos v$

$y = u\sin v$

$$J = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cdot\cos^2 v + u\sin^2 v = u$$

$$\underset{dA}{dx\,dy} \rightsquigarrow J\cdot du\,dv \qquad dA = r\,dr\,d\theta \qquad \rho^2\sin\varphi$$

Change of Variables in a Double Integral: Suppose T is a C^1-transformation whose Jacobian is nonzero & T maps the region S in the uv-plane onto the region R in the xy-plane. Suppose that f is cont. on R & R & S are "type I" or "type II" regions. Suppose T is 1-1 except perhaps on the boundary of S. Then

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv}$$

Absolute value of the determinant

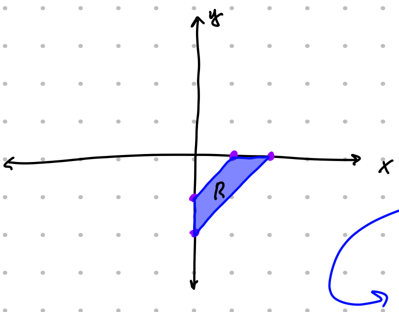
Jacobian of T is a number

$$\int x\cdot e^{x^2}\,dx \qquad u = x^2 \qquad x = u^{1/2}$$

$$\frac{1}{2}\,du = x\,dx \qquad \frac{\partial x}{\partial u} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$$

$$\int \cancel{\sqrt{u}}\, e^u \cdot \frac{1}{2\cancel{\sqrt{u}}}\,du = \int \frac{1}{2}e^u\,du$$

Example: $\iint_R e^{(x+y)/(x-y)}\,dA$ where R is the trapezoid w/ vertices $(1,0)$ $(2,0)$ $(0,-2)$ $(0,-1)$



$e^{(x+y)/(x-y)}$

$e^{u/v}$

u & v

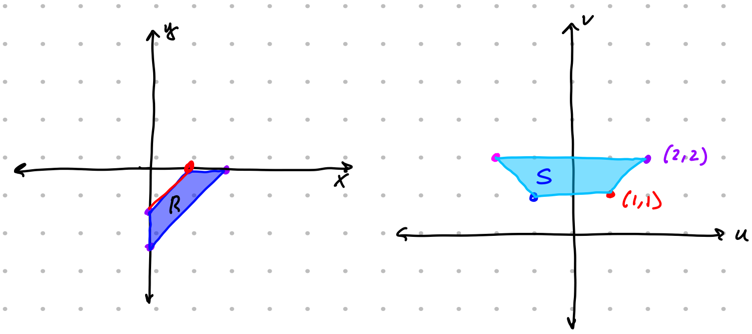
$$\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\}$$ Turn these into $x = g(u,v)$, $y = h(u,v)$

Linear: Maps straight lines to straight lines.

$$\boxed{\begin{aligned} u+v = 2x &\Rightarrow x = \tfrac{1}{2}(u+v) \\ u-v = 2y &\Rightarrow y = \tfrac{1}{2}(u-v) \end{aligned}}$$

* Calculate Jacobian $\quad J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$

$(1,0)$ $(2,0)$ $(0,-2)$ $(0,-1)$

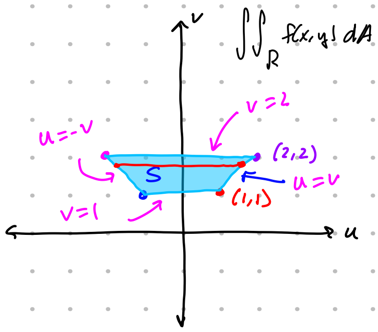


$u(1,0) = 1 \qquad v(1,0) = 1$

$u(2,0) = 2 \qquad v(2,0) = 2$

$u(0,-2) = -2 \qquad v(0,-2) = 2$

$u(0,-1) = -1 \qquad v(0,-1) = 1$



$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\} \qquad \begin{aligned} u+v &= 2x \Rightarrow x = \tfrac{1}{2}(u+v) \\ u-v &= 2y \Rightarrow y = \tfrac{1}{2}(u-v) \end{aligned}$$

$$\iint_R e^{(x+y)/(x-y)}\,dA = \iint_S e^{u/v} \cdot \tfrac{1}{2}\,du\,dv$$

$$\tfrac{1}{2}\int_1^2 \int_{-v}^{v} e^{u/v}\,du\,dv = \tfrac{1}{2}\int_1^2 v e^{u/v}\Big|_{-v}^{v}\,dv$$

$$= \tfrac{1}{2}\int_1^2 v\left(e - \tfrac{1}{e}\right)dv = \tfrac{1}{2}\left(e-\tfrac{1}{e}\right)\int_1^2 v\,dv$$

$$= \tfrac{1}{2}\left(e-\tfrac{1}{e}\right)\tfrac{1}{2}\,3 = \tfrac{3}{4}\left(e-\tfrac{1}{e}\right)$$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\tfrac{1}{2}$$

- * Min/max
- * $\iint$ $\iiint$
- * Change of variable.

Participation Points

1. Let T be a transformation defined by $x = u^2v^3$ and $y = 4 - 2\sqrt{u}$. Compute the Jacobian (determinant)
2. The point P is given in cartesian coordinates by $(1, 1, 1)$. Convert to polar coordinates.