Monday, July 18, 2022
MATH 164 Lecture Notes

Office Hours: 11am-12pm EST

Section 15.6: Triple Integrals

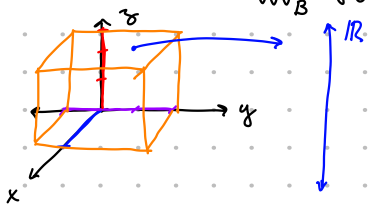
$$\iiint_E \overset{=1}{f(x,y,z)}\,dV$$

- A triple integral is just like a double integral except that the domain of the integrand is in $\mathbb{R}^3$.
- Note that we can't "see" what we are integrating except when the function is 1. In this case, we are finding the volume of a 3-dimensional region, just as integrating $f(x,y)=1$ over $D \subset \mathbb{R}^2$ gives you the area of D.

Fubini's Theorem for Triple Integrals: If f is continuous on the rectangular box $B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

Example: Evaluate $\iiint_B xyz^2\,dV$ where $B=\{(x,y,z) : 0\le x\le 1,\ -1\le y\le 2,\ 0\le z\le 3\}$



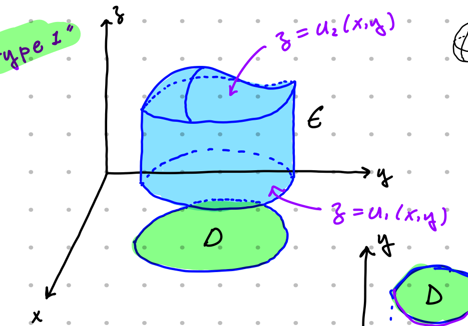
$$\int_0^3\int_{-1}^2\int_0^1 xyz^2\,\overbrace{dx\,dy\,dz}^{dV}$$

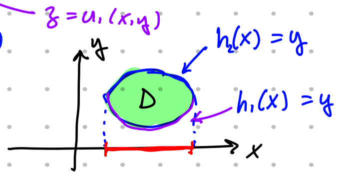
$$\int_0^3\int_{-1}^2 yz^2\,\tfrac{1}{2}\,dy\,dz = \tfrac{1}{2}\int_0^3 z^2\left(\tfrac{4}{2}+\tfrac{1}{2}\right)dz$$

$$\tfrac{5}{4}\int_0^3 z^2\,dz = \tfrac{5}{4}\cdot\tfrac{1}{\cancel{3}}(\overset{9}{\cancel{27}}) = \tfrac{45}{4}$$

We can also integrate over a bounded & closed region $E\subset\mathbb{R}^3$:

"type 1"





$$\iiint_E f(x,y,z)\,dV$$

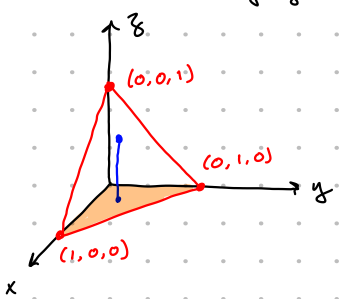
$$=\int_a^b\int_{h_1(x)}^{h_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

functions of 2 or fewer variables

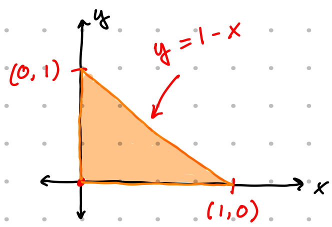
functions of 1 variable or constant

Always constants.

Example: Evaluate $\iiint_E z\,dV$ where E is the solid tetrahedron bounded by the 4 planes $x=0, y=0, z=0$ and $x+y+z=1$.



1. The plane intersects the x-axis when $y=0$ & $z=0$
   $\Rightarrow x=1$
2. The plane intersects the y-axis when $x=0$ & $z=0$
   $\Rightarrow y=1$
3. The plane intersects the z-axis when $x=0$ & $y=0$
   $\Rightarrow z=1$



$$\int_0^1\int_0^{1-x}\int_0^{x+y-1} z\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\tfrac{1}{2}(x+y-1)^2\,dy\,dx$$

$$(x+y-1)(x+y-1) = x^2+y^2+1+xy-x+xy-y-x-y$$
$$= x^2+y^2+2xy-2x-2y+1$$

$$\tfrac{1}{2}\int_0^1\int_0^{1-x}(x^2+y^2+2xy-2x-2y+1)\,dy\,dx$$

3–8 Evaluate the iterated integral.

3. $\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,dx\,dy\,dz$

4. $\int_0^1\int_y^{2y}\int_0^{x+y} 6xy\,dz\,dx\,dy$

5. $\int_1^2\int_0^{2z}\int_0^{\ln x} xe^{-y}\,dy\,dx\,dz$

6. $\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy$

7. $\int_0^\pi\int_0^1\int_0^{\sqrt{1-z^2}} z\sin x\,dy\,dz\,dx$

8. $\int_0^1\int_0^1\int_0^{2-x^2-y^2} xye^z\,dz\,dy\,dx$

5) $\int_1^2\int_0^{2z}\int_0^{\ln(x)} xe^{-y}\,dy\,dx\,dz$

$e^{\ln(A)} = A$ $\quad e^{-\ln(x)} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$

$e^0 =$

$=\int_1^2\int_0^{2z} x\left(-e^{-y}\Big|_0^{\ln(x)}\right)dx\,dz$

$=-\int_1^2\int_0^{2z} x\left(\frac{1}{x}-1\right)dx\,dz$

$=-\int_1^2\int_0^{2z}(1-x)\,dx\,dz = -\int_1^2\left(2z-\frac{1}{2}(4z^2)\right)dz$

$=\int_1^2(2z^2-2z)\,dz = \frac{2}{3}z^3\Big|_1^2 - z^2\Big|_1^2$

9–18 Evaluate the triple integral.

9. $\iiint_E y\,dV$, where $E=\{(x,y,z)\mid 0\leqslant x\leqslant 3, 0\leqslant y\leqslant x, x-y\leqslant z\leqslant x+y\}$

10. $\iiint_E e^{z/y}\,dV$, where $E=\{(x,y,z)\mid 0\leqslant y\leqslant 1, y\leqslant x\leqslant 1, 0\leqslant z\leqslant xy\}$

11. $\iiint_E \frac{z}{x^2+z^2}\,dV$, where $E=\{(x,y,z)\mid 1\leqslant y\leqslant 4, y\leqslant z\leqslant 4, 0\leqslant x\leqslant z\}$

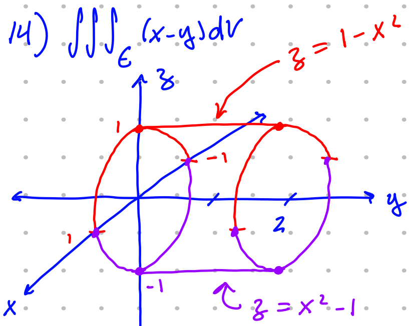
12. $\iiint_E \sin y\,dV$, where E lies below the plane $z=x$ and above the triangular region with vertices $(0,0,0)$, $(\pi,0,0)$, and $(0,\pi,0)$

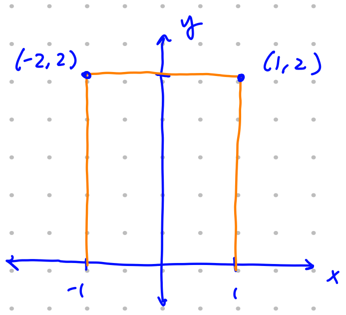
13. $\iiint_E 6xy\,dV$, where E lies under the plane $z=1+x+y$ and above the region in the xy-plane bounded by the curves $y=\sqrt{x}$, $y=0$, and $x=1$

14. $\iiint_E (x-y)\,dV$, where E is enclosed by the surfaces $z=x^2-1$, $\boxed{z=1-x^2}$, $y=0$, and $y=2$

15. $\iiint_T y^2\,dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, and $(0,0,2)$

14) $\iiint_E (x-y)\,dV$





$\int_0^2\int_{-1}^1\int_{x^2-1}^{1-x^2}(x-y)\,dz\,dx\,dy$

10) $\iiint_E e^{z/y}\,dV$ $\quad E=\{(x,y,z): 0\le y\le 1, y\le x\le 1, 0\le z\le xy\}$

$\int_0^1\int_y^1\int_0^{xy} e^{z/y}\,dz\,dx\,dy$

$\int_0^1\int_y^1\left(ye^{z\cdot\frac{1}{y}}\Big|_0^{xy}\right)dx\,dy$

$=\int_0^1\int_y^1\left(ye^{xy/y}-y\right)dx\,dy$

$=\int_0^1\int_y^1(ye^x-y)\,dx\,dy$

$=\int_0^1\left(ye^x\Big|_y^1 - yx\Big|_y^1\right)dy$

$=\int_0^1\left(ye\cdot e^y-(y-y^2)\right)dy$

$=e\int_0^1(y\cdot e^y+y^2-y)\,dy$

$=e\int_0^1 y\cdot e^y\,dy + e\int_0^1 y^2\,dy - e\int_0^1 y\,dy$

Integration by parts

19–22 Use a triple integral to find the volume of the given solid.

19. The tetrahedron enclosed by the coordinate planes and the plane $2x+y+z=4$

20. The solid enclosed by the paraboloids $y=x^2+z^2$ and $y=8-x^2-z^2$

21. The solid enclosed by the cylinder $y=x^2$ and the planes $z=0$ and $y+z=1$

22. The solid enclosed by the cylinder $x^2+z^2=4$ and the planes $y=-1$ and $y+z=4$

$\boxed{\int u\,dv = uv - \int v\,du}$

$dv = e^y \Rightarrow v = e^y$

$u = y \quad du = 1dy$

$\int ye^y = ye^y - \int e^y\,dy$

$= ye^y - e^y$ ✓

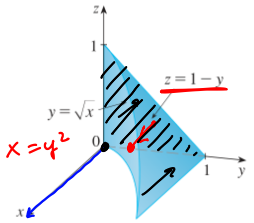
* integration by parts

Exam: Up to 15.9

33. The figure shows the region of integration for the integral

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$$

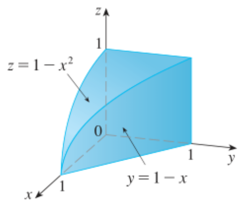
Rewrite this integral as an equivalent iterated integral in the five other orders.



34. The figure shows the region of integration for the integral

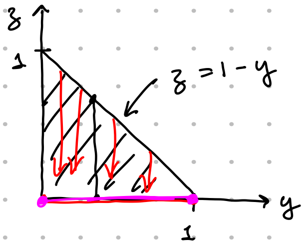
$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



$$\boxed{\int e^{x^2}dx}$$

$$\int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

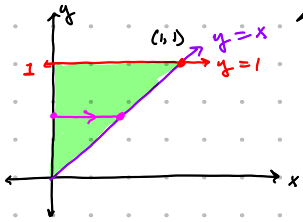


Example: Sometimes you have to change the order of integration

Integrate: $\int_0^1\int_x^1 e^{y^2}dy\,dx = \int_0^1\int_0^y e^{y^2}dx\,dy$

$= \boxed{\int_0^1 e^{y^2}\cdot y\,dy}$

u-sub or integration by parts



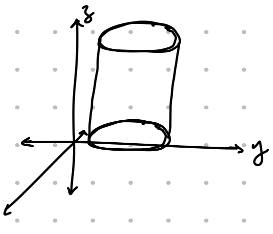
impossible

$u = y^2$

$\frac{1}{2}du = y\,dy$

$\frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\cdot(e-1) = \boxed{\frac{e}{2}-\frac{1}{2}}$

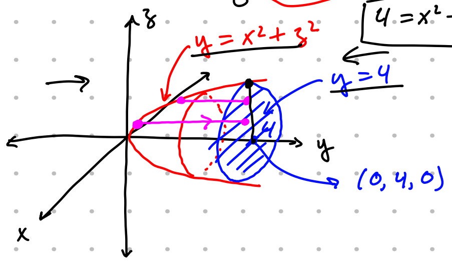
Section 15.7: Triple Integrals in Cylindrical Coordinates (polar)



* Cylindrical coordinates: $x = r\cos\theta$, $y = r\cos\theta$, $z = z$

Example: $\iiint_E \sqrt{x^2+z^2}\, dV$ ($\sqrt{x^2+z^2} \to r$) E is the region bounded by $y = x^2+z^2$ + the plane $y = 4$

$\boxed{4 = x^2+z^2}$

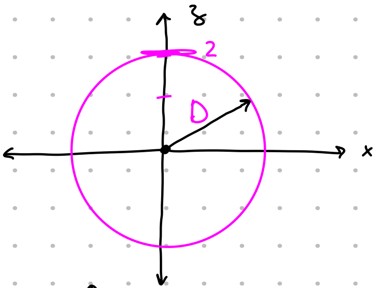


$x = r\cos\theta$ $\quad \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r$

$z = r\sin\theta$

$\boxed{\text{don't forget } dA = r\,dr\,d\theta}$

$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\, dy\, dr\, d\theta$$



$$= \int_0^{2\pi}\int_0^2 r^2(4 - r^2)\, dr\, d\theta$$

r^4

$$\boxed{= 2\pi\left(\frac{4}{3}(2^3) - \frac{1}{5}(2^5)\right)}$$

$x \geq 0$ $\quad y \geq 0$ $\quad + z \geq 0$

19. Evaluate $\iiint_E (x + y + z)\, dV$, where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

20. Evaluate $\iiint_E (x - y)\, dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane, and below the plane $z = y + 4$.

21. Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

22. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

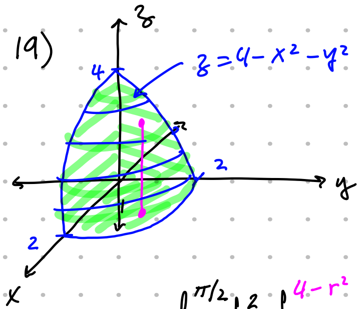
23. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

24. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

25. (a) Find the volume of the region E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

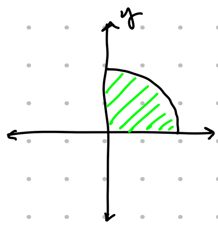
(b) Find the centroid of E (the center of mass in the case where the density is constant).

19)



$0 = 4 - x^2 - y^2$

$x^2 + y^2 = 4$



$x = r\cos\theta$

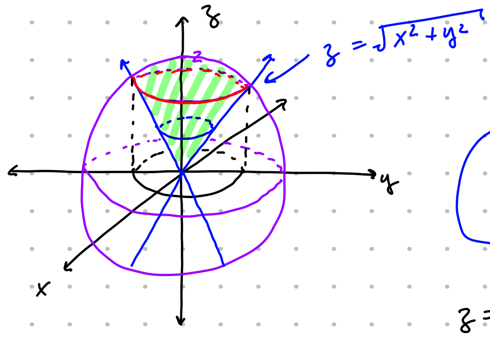
$y = r\sin\theta$

$$\int_0^{\pi/2}\int_0^2\int_0^{4-r^2} r(r\cos\theta + r\sin\theta + z)\, dz\, dr\, d\theta$$

$z = \sqrt{2 - x^2 - y^2} \rightsquigarrow \sqrt{2 - r^2}$

$\boxed{z^2 = x^2 + y^2}$

23)



$x = r\cos\theta$ $\quad dV = r\,dz\,dr\,d\theta$

$y = r\cos\theta$

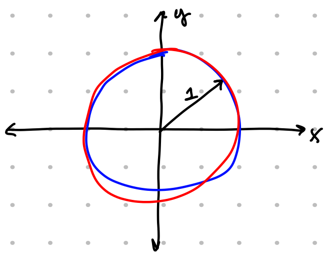
$$2\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

$\sqrt{4} = \pm 2$

$\left.\begin{aligned} z &= \sqrt{x^2+y^2} \\ z &= \sqrt{2 - x^2 - y^2} \end{aligned}\right\}$ $\quad x^2 + y^2 = 2 - x^2 - y^2$

$2(x^2 + y^2) = 2$

$x^2 + y^2 = 1$



Participation Points

1. Set up but do not solve the triple integral representing the volume bounded by the plane $z = 0$ and the paraboloid $-x^2 - y^2 + 9 = z$.