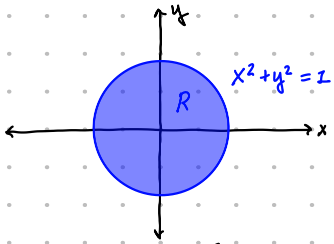
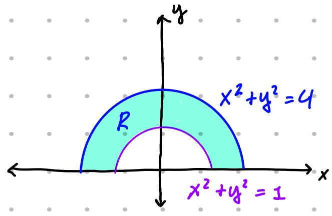
Thursday, July 14, 2022

MATH 164 Lecture Notes

Section 15.3: Double Integrals in Polar Coordinates

In addition to rectangles, we have special methods to integrate over polar regions.

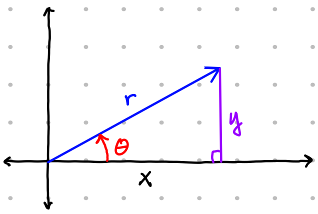




$0 \le \theta \le 2\pi$

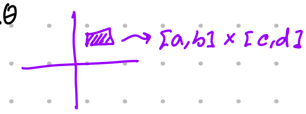
$R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$R(r,\theta) = \{(r,\theta) : 1 \le r \le 4,\ 0 \le \theta \le \pi\}$

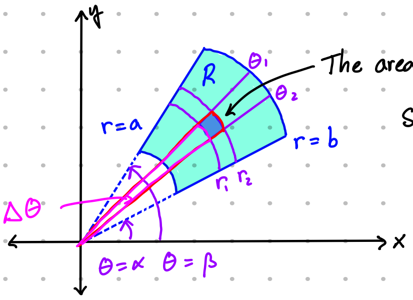


• A polar rectangle $R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$

$$\int_a^b \int_c^d f(x,y)\, dA \qquad dA \to r\,dr\,d\theta$$



$dx\,dy \quad dy\,dx$



The area of a sector of a circle is $\frac{1}{2}r^2\theta$.

$\theta = X\,2\pi \qquad X = \frac{\theta}{2\pi}$

$A = X \cdot \pi r^2$

$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{\theta r^2}{2}$

So the infinitessimal area is:

$$\tfrac{1}{2}r_1^2\Delta\theta - \tfrac{1}{2}r_2^2\Delta\theta = \tfrac{1}{2}\Delta\theta(r_1^2 - r_2^2) = \tfrac{1}{2}\Delta\theta\,(r_1+r_2)(r_1-r_2)$$

$\frac{1}{2}(r_1+r_2) = r^*$ is the average radius.

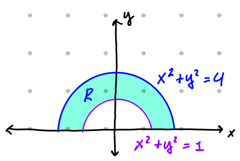
$$= r^*\Delta r\Delta\theta = \Delta A$$

$\therefore dA = r\,dr\,d\theta$

=> To integrate a function $f(x,y)$ over a polar rectangle: • A polar rectangle $R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$

$$\boxed{\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Example: Evaluate $\iint_R (3x+4y^2)\,dA$ where R is the region in the upper plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$R(r,\theta) = \{(r,\theta) : 1 \le r \le 4,\ 0 \le \theta \le \pi\}$

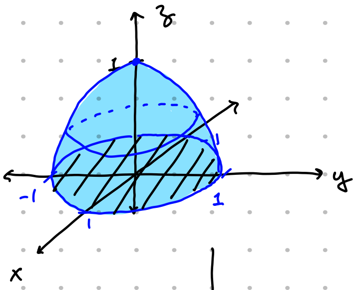
$$\int_0^\pi \int_1^4 3r^2\cos\theta\, dr\,d\theta + \int_0^\pi \int_1^4 4r^3\sin^2\theta\, dr\,d\theta$$

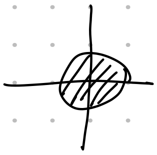
$$= \int_0^\pi \cancel{3}\cos\theta\, \frac{1}{\cancel{3}} r^3 \Big|_1^4 d\theta + \int_0^\pi \cancel{4}\sin^2\theta\, \frac{1}{\cancel{4}} r^4 \Big|_1^4 d\theta$$

$$= \int_0^\pi \cos\theta\,(4^3-1)\, d\theta + \int_0^\pi (4^4-1)\sin^2\theta\, d\theta$$

Example: Find the volume of a solid bounded by the plane $\overset{\text{xy-plane}}{z=0}$ and the paraboloid $z = 1-x^2-y^2$.

+ $z=0$





$$\iint_R f(x,y)\,dA$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

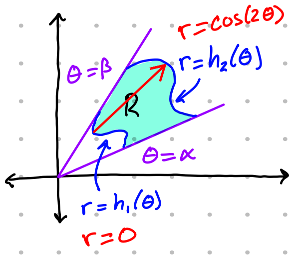
$$f(x,y) = 1-x^2-y^2$$

$$f(r,\theta) = 1-r^2$$

$$\int_0^{2\pi}\int_0^1 \underbrace{(1-r^2)\overbrace{r\,dr\,d\theta}^{dA}}_{r-r^3} = \int_0^1 2\pi r(1-r^2)\,dr$$

$$= \int_0^{2\pi}\left(\tfrac{1}{2}-\tfrac{1}{4}\right)d\theta = \tfrac{1}{4}\cdot 2\pi = \tfrac{1}{2}\pi$$

* We can also integrate over regions bounded between polar curves:

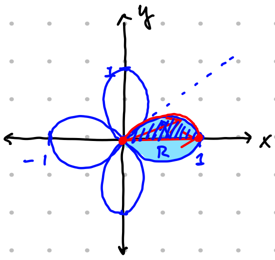


$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

integrat $\iint_R \underbrace{r\,dr\,d\theta}_{dA}$

$r=\cos(k\theta)$ — k is even, then the rose has $2k$ petles; k is odd it has k petles.

Example: Use a double integral to find the area enclosed by 1 loop of the 4-leaved rose $r=\cos(2\theta)$.

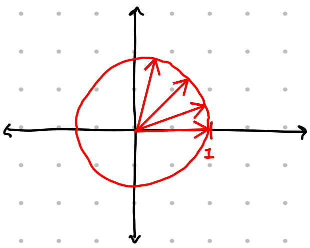


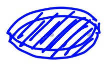
θ	$\cos(2\theta)=r$
0	$\cos(0)=1$
$\pi/4$	$\cos(\pi/2)=0$
$\pi/2$	$\cos(\pi)=1$

$$A = 2\int_0^{\pi/4}\int_0^{\cos(2\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4} \tfrac{1}{2}\cos^2(2\theta)\,d\theta$$

$\sin(2a)=2\sin(a)\cos(a)$
$\cos(2a)=\cos^2(a)-\sin^2(a)$
$\cos(2a)=2\cos^2(a)-1$
$\cos(2a)=1-2\sin^2(a)$
$\tan(2a)=\dfrac{2\tan(a)}{1-\tan^2(a)}$





$$= 2\int_0^{\pi/4}\cos^2(2\theta)\,d\theta$$

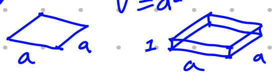
$2\cos^2(a)=\cos(2a)+1$

$\cos^2(a)=\frac{1}{2}\cos(2a)+\frac{1}{2}$

$$= 2\int_0^{\pi/4}\left(\tfrac{1}{2}\cos(4\theta)+\tfrac{1}{2}\right)d\theta$$

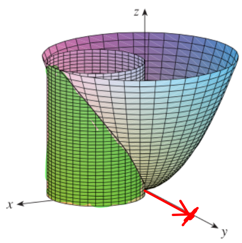
$\cos^2(2\theta)=\frac{1}{2}\cos(4\theta)+\frac{1}{2}$

$A=a^2$ $\quad V=a^2$

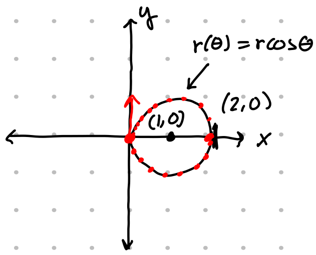


$$= 2\Big(\underbrace{\tfrac{1}{8}\sin(4\theta)\Big|_0^{\pi/4}}_{0} + \underbrace{\tfrac{1}{4}\theta\Big|_0^{\pi/4}}_{2\cdot\frac{\pi}{16}}\Big) = \frac{\pi}{8}$$

Example: Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above the xy-plane, and inside the cylinder $x^2+y^2=2x$.



$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



$f(x,y)=x^2+y^2 \qquad f(r,\theta)=r^2$

$R=\{(r,\theta): 0\le r\le 2\cos\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\}$

$x^2+y^2=2x$

$$\text{Volume} = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$2\cos(\pi/4)\neq 0$

$2\cos(-\pi/2)$

$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$

$r = 2\cos\theta$

$$\text{Volume} = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{4}2^4\cos^4\theta\,d\theta = \frac{2^5}{2^2}\int_0^{\pi/2}\underbrace{\cos^4\theta}_{(\cos^2\theta)^2}\,d\theta$$

$\left(\frac{1}{2}\cos(2a)+\frac{1}{2}\right)^2$

$\cos^2(a)=\frac{1}{2}\cos(2a)+\frac{1}{2}$

15–18 Use a double integral to find the area of the region.

15. One loop of the rose $r=\cos 3\theta$

16. The region enclosed by both of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$

17. The region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$

18. The region inside the cardioid $r=1+\cos\theta$ and outside the circle $r=3\cos\theta$

19–27 Use polar coordinates to find the volume of the given solid

19. Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leqslant 25$

20. Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leqslant x^2 + y^2 \leqslant 4$

21. Below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \leqslant 1$

22. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

23. A sphere of radius a

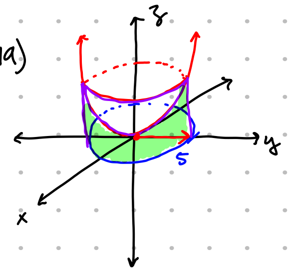
24. Bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant

25. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

26. Bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$

27. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

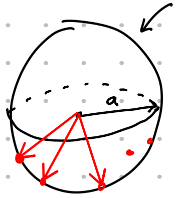
19)



$$\int_0^{2\pi}\int_0^5 r^3\,dr\,d\theta$$

$$= 2\pi\,\frac{5^4}{4}$$

23)



$x^2+y^2+z^2=a^2$

$z = \sqrt{a^2-x^2-y^2}$

$$2\int_0^{2\pi}\int_0^a \left(\sqrt{a^2-r^2}\; r\right) dr\,d\theta$$

$u = a^2 - r^2$

$du = -2r\,dr$

$$2\pi\int_0^{a^2}\sqrt{u}\,du$$

Break: X:05

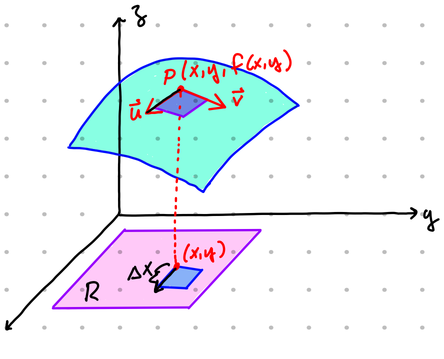
Section 15.5: Surface Area

Area(parallelogram) $= |\vec{u} \times \vec{v}|$ parallel to the x-axis / parallel to the y-axis

* Skipping 15.4 (Applications)

$\vec{u} = \langle \Delta x, 0, f_x(x,y)\Delta x \rangle$

$\vec{v} = \langle 0, \Delta y, f_y(x,y)\Delta y \rangle$



- An infinitesimal surface area element has area $|\vec{u} \times \vec{v}|$ where

  $\vec{u} = \langle \Delta x, 0, f_x(x_i, y_i)\Delta x \rangle$

  $\vec{v} = \langle 0, \Delta y, f_y(x_i, y_i)\Delta y \rangle$

- $|\vec{u} \times \vec{v}| = \sqrt{f_x(x_i,y_i)^2 + f_y(x_i,y_i)^2 + 1}\ \Delta A$

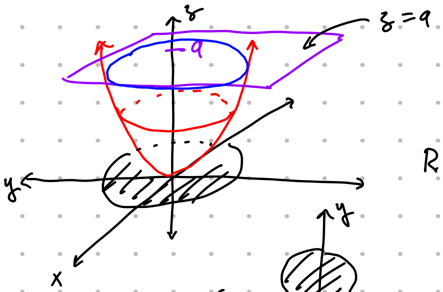
Therefore, the area of the surface with equation $z = f(x,y)$, $(x,y) \in D$ where f_x and f_y are continuous, is:

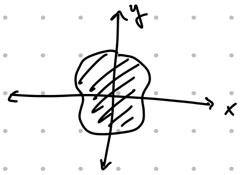
$$A(S) = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\ dA$$

$$= \iint_D \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\ dA$$

$9 = r^2 \quad r = 3$

Example: Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.





$$\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

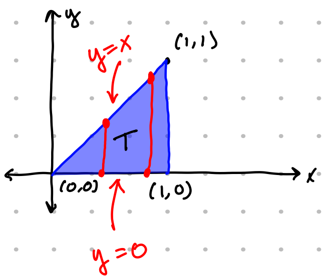
$R =$

$z = x^2 + y^2$

$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$

$\sqrt{1 + 4x^2 + 4y^2} = \sqrt{1 + 4r^2}$

Example: Find the surface area of the portion of the surface $z = x^2 + 2y$ that lies above the triangle T.



$$A(S) = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\ dA$$

$$= \iint_D \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\ dA$$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = 2$

u-sub integral

$$\int_0^1\int_0^x \sqrt{4x^2+5}\ dy\, dx = \int_0^1 \sqrt{4x^2+5}\ x\, dx$$

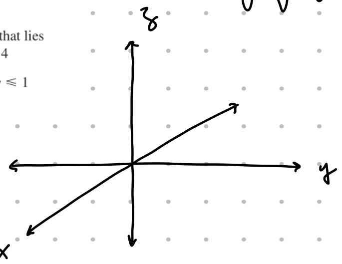
1–12 Find the area of the surface.

1. The part of the plane $5x + 3y - z + 6 = 0$ that lies above the rectangle $[1, 4] \times [2, 6]$
2. The part of the plane $6x + 4y + 2z = 1$ that lies inside the cylinder $x^2 + y^2 = 25$
3. The part of the plane $3x + 2y + z = 6$ that lies in the first octant
4. The part of the surface $2y + 4z - x^2 = 5$ that lies above the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 4)$
5. The part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $z = -2$
6. The part of the cylinder $x^2 + z^2 = 4$ that lies above the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$
7. The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$
8. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

3) Find the surface area of the part of $3x + 2y + z = 6$ that lies in the first octant

$$z = 6 - 3x - 2y$$

$$\iint \sqrt{9 + 4 + 1}\, dA = \boxed{\sqrt{14} \int_0^2 \int_0^{-\frac{3}{2}x + 3} dy\, dx}$$



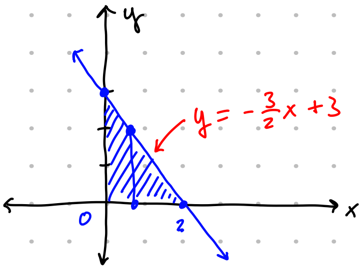
$$R = \{(x, y) : x \geq 0,\ y \geq 0, \&\ z \geq 0\}$$

$$z \geq 0 \iff 6 - 3x - 2y \geq 0$$

$$\iff 6 \geq 3x + 2y$$

$$6 - 3x \geq 2y$$

$$3 - \frac{3}{2}x \geq y$$



Participation Points

1. Set up (but do not solve) the double integral representing the area of one leaf enclosed by the curve $r = \cos(3\theta)$.