Wednesday, July 13, 2022

MATH 164 Lecture Notes

Section 14.8: The Method of Lagrange Multipliers

- Short review:
  - For $f(x,y,z)$, $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$ ← gradient vector
  - 2^{nd} derivative test
  - local + global max/min problems
  - tangent planes

Example: Find the absolute max + absolute min of $f(x,y) = 192x^3 + y^2 - 4xy^2$ on the triangle with vertices $(0,0)$, $(4,2)$ and $(-2,2)$

Step 1: Find f_x + f_y + set $=0$

$f_x = 576x^2 - 4y^2 \qquad 576x^2 - 4y^2 = 0$

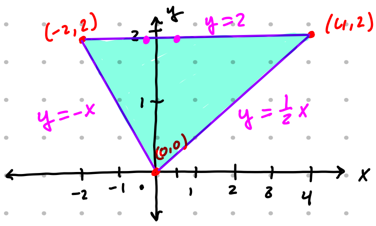
$f_y = 2y - 8xy \qquad 2y - 8xy = 0 \Rightarrow y(1-4x) = 0 \Rightarrow y = 0$ or $x = 1/4$

Case 1: when $y=0$, $576x^2 = 0 \Rightarrow x = 0 \Rightarrow (0,0)$

Case 2: when $x = 1/4$, $576/16 - 4y^2 = 0 \Rightarrow 36 - 4y^2 = 0 \Rightarrow 4y^2 = 36 \Rightarrow \pm 2y = 6 \Rightarrow y = \pm 3$

$\Rightarrow (\frac{1}{4}, 3)$, $(\frac{1}{4}, -3)$ ⤳ Not contained in the triangle (ignore them).

Step 2: Check boundary



When $y=-x$, $-2 \le x \le 0$ $\qquad f(x,y) = 192x^3 + y^2 - 4xy^2$

$g(x) = f(x,-x) = x^2 + 188x^3$

$g'(x) = 2x + 564x^2 = 0 \Rightarrow x(1+282x) = 0 \Rightarrow x=0$ or $x = -\frac{1}{282}$

$(0,0) \qquad (-\frac{1}{282}, \frac{1}{282})$

$\boxed{f(0,0) = 0} \qquad \boxed{f(-\frac{1}{282}, \frac{1}{282}) = \frac{1}{238,572}} \qquad \boxed{f(-2,2) = -1500}$ absolute min

When $y=2$, $-2 \le x \le 4$

$g(x) = f(x,2) = 192x^3 - 16x + 4$

$g'(x) = 576x^2 - 16 \Rightarrow x = \pm 1/6$

$(\frac{1}{6}, 2) \quad (-\frac{1}{6}, 2)$

$\boxed{f(\frac{1}{6}, 2) = \frac{20}{9}} \qquad \boxed{f(-\frac{1}{6}, 2) = \frac{52}{9}} \qquad \boxed{f(4,2) = 12{,}228}$ absolute max

When $y = \frac{1}{2}x$, $0 \le x \le 4$

$g(x) = f(x, \frac{1}{2}x) = \frac{1}{4}x^2 + 191x^3$

$g'(x) = \frac{1}{2}x + 573x^2 = x\left(\frac{1}{2} + 573x\right) = 0 \Rightarrow x = 0$ or $x = -\cancel{\frac{1}{1146}}$ ignore.

$(0,0)$ No new information.

Step 3: Compare

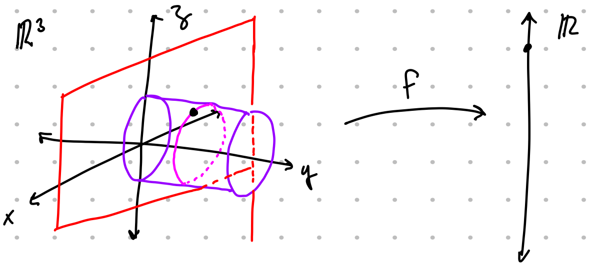
Example: Lagrange Multipliers w/ multiple constraints

Theorem: For $f(x,y,z)$ subject to the constraints $g(x,y,z)=k$, $h(x,y,z)=c$, the critical points are the solutions to:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \qquad \nabla f = \lambda \nabla g$$
$$g(x,y,z)=k$$
$$h(x,y,z)=c$$

$g(x,y,z) = 4x-3y \qquad h(x,y,z) = x^2+z^2$

- Find the max/min values of $f(x,y,z) = 3x^2+y$ subject to the constraints $4x-3y=9$ and $x^2+z^2=9$



- $\nabla f(x,y,z) = \langle 6x, 1, 0\rangle$
- $\nabla g(x,y,z) = \langle 4, -3, 0\rangle$
- $\nabla h(x,y,z) = \langle 2x, 0, 2z\rangle$

System of equations: $\nabla f = \lambda \nabla g + \mu \nabla h$

1. $6x = 4\lambda + 2\mu x \rightsquigarrow 6x = -\frac{4}{3} + 2\mu x \rightsquigarrow x(6-2\mu) = -\frac{4}{3}$
2. $1 = -3\lambda \rightsquigarrow \lambda = -\frac{1}{3}$
3. $0 = 2\mu z \rightsquigarrow \mu = 0$ or $z=0$
4. $4x-3y=9$
5. $x^2+z^2=9$

Case 1: $z=0 \quad 5 \Rightarrow x = \pm 3$

Case 1.1: $z=0,\ x=3 \quad 4 \Rightarrow 12-3y=9 \Rightarrow -3y=-3 \Rightarrow \boxed{y=1}$

Case 1.2: $z=0\ x=-3 \quad 4 \Rightarrow -12-3y=9 \Rightarrow -3y=21 \Rightarrow \boxed{y=-7}$

$\boxed{\begin{matrix}(3,1,0)\\(-3,-7,0)\end{matrix}}$

Case 2: $\mu=0 \quad 1 \Rightarrow 6x = -\frac{4}{3} \quad x = -\frac{4}{18} = -\frac{2}{9}$

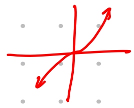
$-\frac{8}{9} - 3y = 9 \quad -8-27y=81 \Rightarrow -27y = 89 \Rightarrow \boxed{y = -\frac{89}{27}} \qquad z = \pm\frac{5\sqrt{29}}{9}$

$\boxed{\left(-\frac{2}{9}, -\frac{89}{27}, \frac{5\sqrt{29}}{9}\right)} \boxed{\left(-\frac{2}{9}, -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right)}

Example: Let $\boxed{f(x,y) = x^4 + y^4 - x^3}$. Find and classify the critical points

$f_x(x,y) = 4x^3 - 3x^2 = 0 \Rightarrow x^2(4x-3) \Rightarrow x = 0$ or $4x = 3 \Rightarrow x = 3/4$

$f_y(x,y) = 4y^3 = 0 \Rightarrow y = 0$



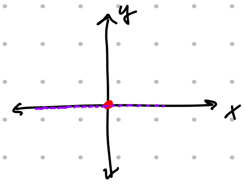
Critical points: $(0,0)$, $(3/4, 0)$

$f_{xx}(x,y) = 12x^2 - 6x$

$f_{xy}(x,y) = 0$

$f_{yy}(x,y) = 12y^2$

$$D(0,0) = \begin{vmatrix} 12x^2 - 6x & 0 \\ 0 & 12y^2 \end{vmatrix} = 0 \Leftrightarrow 2^{nd} \text{ derivative test is inconclusive}$$



$$D(3/4, 0) = \begin{vmatrix} \frac{12 \cdot 9}{16} - \frac{18}{4} & 0 \\ 0 & 0 \end{vmatrix} = 0$$

Is $(0,0)$ a min, max, or saddle point of f?

when $y = 0$

$g(x) = f(x,0) = x^4 - x^3$

when $x^4 < x^3$, negative

$x^4 > x^3$, positive

$\Rightarrow$ inflection point

$\frac{1}{a^4} < \frac{1}{a^3}$

when $x < 0$, $x^4 > 0$

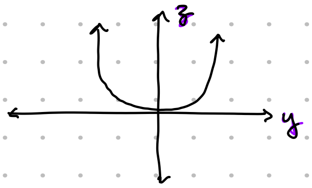
$x^3 < 0$

$x > 0$

$\left(\frac{1}{2}\right)^4 = \frac{1}{2^4} - \frac{1}{2^3} < 0$

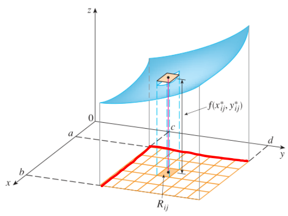
when $x = 0$

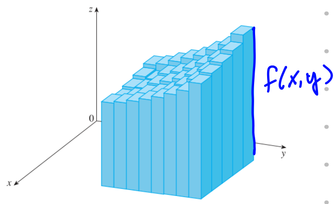
$z = g(y) = f(0,y) = y^4 \Rightarrow (0,0)$ is min



$\therefore$ We know $(0,0)$ is a saddle point of $f(x,y)$.

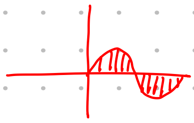
Section 15.1: Double Integrals over Rectangles



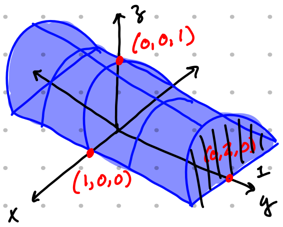


- If $f(x,y) \geq 0$, then the volume V of the solid that above the rectangle R + below the surface $z = f(x,y)$ is

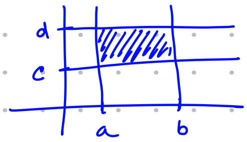
$$\boxed{V = \iint_R f(x,y)\,dA}$$

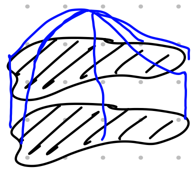


Example: $R = \{(x,y) : -1 \leq x \leq 1,\ -2 \leq y \leq 2\}$. Evaluate $\iint_R \sqrt{1-x^2}\,dA$



$$V = \iint_R \sqrt{1-x^2}\,dA = \frac{\pi}{2} \cdot 4 = 2\pi$$





▫ Iterated Integrals: If $R = [a,b] \times [c,d]$ and $f(x,y)$ is continuous/integrable,

$$V = \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Fubini's Theorem: If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

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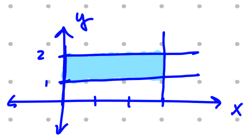
Example: $\int_0^3 \left(\int_1^2 x^2 y\,dy \right) dx$

$f(x,y) = x^2 y$



treat x as constant

$$\int_0^3 \left(\int_1^2 x^2 y\,dy \right) dx = \int_0^3 \left(x^2 \int_1^2 y\,dy \right) dx = \int_0^3 x^2 \left(\frac{1}{2} y^2 \Big|_1^2 \right) dx$$

$$= \int_0^3 x^2 \left(\frac{3}{2} \right) dx = \frac{3}{2} \int_0^3 x^2\,dx = \cancel{\frac{3}{2}} \cdot \frac{1}{\cancel{3}} x^3 \Big|_0^3 = \frac{27}{2}$$

$$x^2 y = \overset{x^2}{f(x)}\,\overset{y}{g(y)}$$

$$\iint_R f(x)\,g(y)\,dA = \int_0^3 x^2\,dx \cdot \int_1^2 y\,dy$$

$$\frac{\partial}{\partial x}\left(-\frac{1}{y} \cos(xy) \right) = \sin(xy)$$

Example: $\iint_R y \cdot \sin(xy)\,dA \quad R = [1,2] \times [0,\pi]$

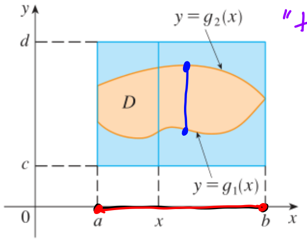
$$\int_0^\pi \int_1^2 y \sin(xy)\,dx\,dy = \int_0^\pi y \int_1^2 \sin(xy)\,dx\,dy$$

$$= \int_0^\pi \cancel{y} \left(-\frac{1}{\cancel{y}} \cos(xy) \Big|_1^2 \right) dy$$

$$= -\int_0^\pi \left(\cos(2y) - \cos(y) \right) dy$$

$$= -\frac{1}{2} \sin(2y) \Big|_0^\pi + \sin(y) \Big|_0^\pi = 0$$

Section 15.2: Double Integrals over General Regions



"type 1"

If f is continuous on a type 1 region D such that

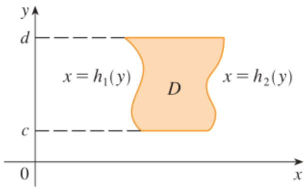
$$D = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx \qquad \text{I}$$

functions on the inside

constants on the outside.

"type 2"



II

If f is continuous on a type 2 region D such that

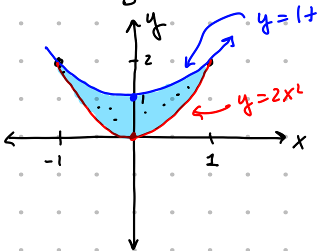
$$D = \{(x,y) \in \mathbb{R}^2 : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

functions on the inside

constants on the outside.

Evaluate: $\iint_D (x+2y)\, dA$ where D is the region bounded between $y = 2x^2$ and $y = 1+x^2$



★ Find out where the bounds intersect

$$2x^2 = 1 + x^2$$

$$x^2 = 1 \implies x = \pm 1 \qquad (1,2) \ + \ (-1,2)$$

$$\int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx = \int_{-1}^{1} \left(x\cdot y + y^2\right)\Big|_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \left(x(1+x^2) + (1+x^2)^2 - x(2x^2) - 4x^4\right) dx$$

$$= \int_{-1}^{1} \left(x + \cancel{x^3} + \cancel{x^4} + 2x^2 + 1 - \cancel{2x^3} - \cancel{4x^4}\right) dx$$

$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right) dx$$

Example: Find the volume of the shape bounded by the planes

$$\begin{cases} x + 2y + z = 2 \\ x = 2y \\ x = 0 \\ z = 0 \end{cases}$$

$2y = 2$

$y = \frac{1}{2}x$

$4y = 2$

$y = 2$

$x = 4$

$0 = 2 - x - 2y$

$z = 2 - x - 2y$

$x + 2y = 2$

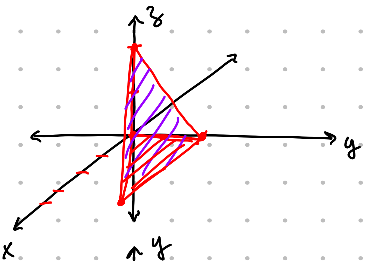
$2y = 2 - x$

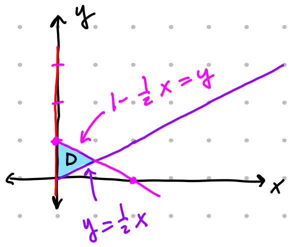
$y = 1 - \frac{1}{2}x$

$2 = x + 2y$

$2y = 2 - x$

$\boxed{y = 1 - \frac{1}{2}x}$





$$\iint_D f(x,y)\, dA$$

$$\iint_D (2 - x - 2y)\, dA$$

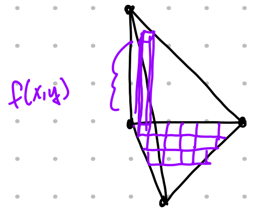
$1 - \frac{x}{2} = \frac{x}{2}$

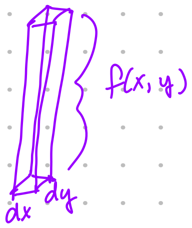
$1 = x$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx$$

$$= \int_0^1 \left(2y - xy - y^2\right)\Big|_{x/2}^{1-x/2} dx = \int_0^1 \Big(2 - x - x + \frac{x^2}{2} - (1 - x/2)^2\Big)$$

$$- \left(x - \frac{x^2}{2} - \frac{x^2}{4}\right) dx$$



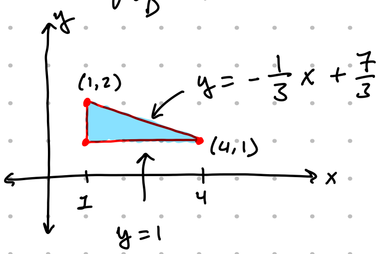


$$\iint f(x,y)\,dx\,dy$$

25) Section 15.2

Example: Find the volume under the surface $z = xy$ & above the triangle w/ vertices $(1,1)$, $(4,1)$, $(1,2)$

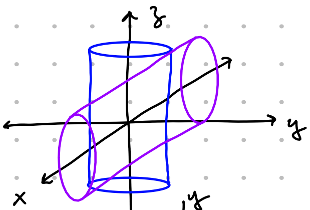
$$\iint_D xy \, dA$$

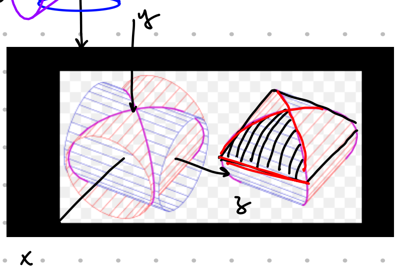


$$\int_1^4 \int_1^{-\frac{1}{3}x + \frac{7}{3}} xy \, dy \, dx$$

$$y = \sqrt{4 - z^2} = f(x,z)$$

Example: Bounded by the cylinders $x^2 + y^2 = 4$ & $y^2 + z^2 = 4$





Find out where they intersect:

$$x^2 + y^2 = y^2 + z^2 \qquad x^2 = z^2 \qquad x = \pm z$$

$$\iint_D \sqrt{4 - z^2} \, dA$$

$$= \int_{-2}^{0} \int_{-z}^{z} \sqrt{4 - z^2} \, dx \, dz$$