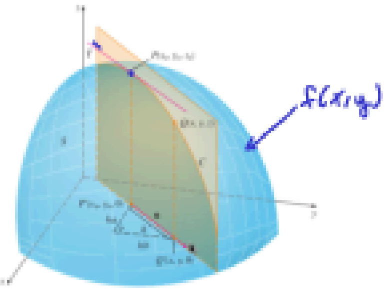
Monday June 7, 2021

MTH 164 Lecture Notes

Section 14.6: Directional Derivatives and Gradient Vectors

$f_x(x,y)$

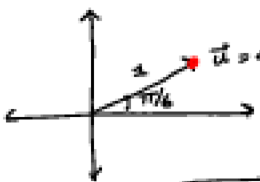


$f(x,y)$

Theorem: $D_{\vec{u}}f(x,y) = \underline{f_x(x,y)}\,a + \underline{f_y(x,y)}\,b$ $\qquad \dfrac{\vec{v}}{|\vec{v}|}$

where $\vec{u} = \langle a, b\rangle$ $\quad$ $\boxed{\vec{u} \text{ must be a unit vector}}$

Example: Find the directional derivative $D_{\vec{u}}f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$, $\vec{u}$ is the unit vector given by $\theta = \pi/6$. What is $D_{\vec{u}}f(1,2)$?



$\vec{u} = \langle \cos(\pi/6), \sin(\pi/6)\rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

$f_x(x,y) = 3x^2 - 3y \qquad f_x(1,2) = 3 - 6 = -3$

$f_y(x,y) = -3x + 8y \qquad f_y(1,2) = -3 + 16 = 13$

$$\boxed{D_{\vec{u}}f(1,2) = -3\cdot\frac{\sqrt{3}}{2} + 13\cdot\frac{1}{2}}$$

Definition: If $f(x,y)$ is a function, then the $\underline{\text{gradient}}$ of f to be the vector function ∇f:

"nabla"

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$\Rightarrow \boxed{D_{\vec{u}}f(x,y) = \nabla f(x,y)\cdot\vec{u}}$$

Example: $f(x,y,z) = x\cdot\sin(yz)$ $\qquad \vec{v} = \langle 1,2,-1\rangle$ @ $(1,3,0)$

$\nabla f = \langle \sin(yz), z\,x\cos(yz), y\,x\cos(yz)\rangle$

$\nabla f(1,3,0) = \langle 0, 0, 3\rangle$

$|\vec{v}| = \sqrt{6}$

$\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt{6}}\langle 1,2,-1\rangle = \vec{u}$

$$\boxed{D_{\vec{u}}f(1,3,0) = \langle 0,0,3\rangle\cdot\frac{1}{\sqrt{6}}\langle 1,2,-1\rangle}$$

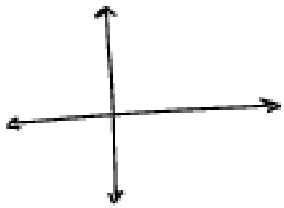
☐ Maximizing the Directional derivative.

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = |\nabla f| \cdot |\vec{u}| \cdot \cos\theta$$
$$= |\nabla f| \cdot \cos\theta$$

When is this at a max?

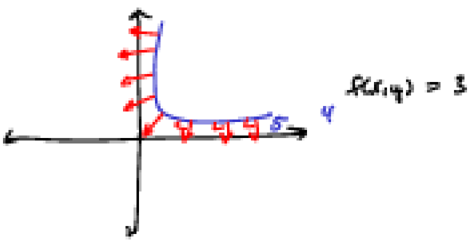
$\theta = 0$

$$\frac{\nabla f(a,b)}{|\nabla f|}$$

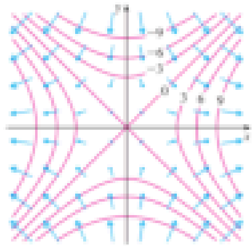


theorem: Suppose f is a diff. function of 2 or 3 variables. The max value of $D_{\vec{u}}f(\vec{x})$ is $|\nabla f(\vec{x})|$ & it occurs when $\vec{u}$ is in the direction of ∇f.

☐ tangent planes to level surfaces.



* tangent plane to the level surface $F(x,y,z) = k$ at $P(x_0, y_0, z_0)$ is perpendicular to $\nabla F(x_0, y_0, z_0)$.



Tangent Plane of $F(x,y,z) = k$ is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$
$$= \nabla F(x_0,y_0,z_0) \cdot (\vec{r} - \vec{r}_0)$$

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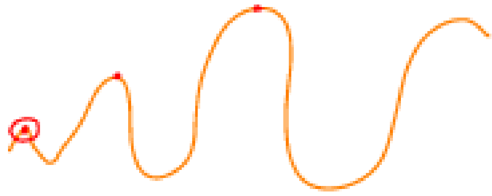


Section 14.7: Min/max problems

Definition: A function $f(x,y)$ has a local max (min) at (a,b) if $f(x,y) \le (\ge) f(a,b)$ when (x,y) is near (a,b). $f(a,b)$ is called a max (min) value.

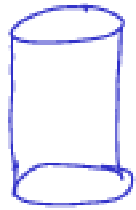
Theorem: If f has a local maximum or minimum at (a,b) + the first order partial derivatives exist, then $f_x(a,b) = f_y(a,b) = 0$

- Find the extreme values of $f(x,y) = y^2 - x^2$

Example: • Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Find its local min/maxes.

$$\left.\begin{aligned} f_x(x,y) &= 2x - 2 = 0 \\ f_y(x,y) &= 2y - 6 = 0 \end{aligned}\right\} \quad \begin{aligned} x &= 1 \\ y &= 3 \end{aligned}$$

$(1,3)$



$f_{xx} = 2$

$f_{yy} = 2$

local min: $(1,3)$
no local max.

$z = x^2 - 2x + 14$

$z = y^2 - 6y$

$$f(1,3) = 1 + 9 - 2 - 18 + 14 = 10 - 20 + 14 = 4$$

The 2nd derivative Test: Suppose the 2nd partial derivatives of f are continuous on a disk with center (a,b) + suppose that $f_x(a,b) = f_y(a,b) = 0$ [that is, (a,b) is a critical point]. Let

*Watch video later.

Hessian matrix

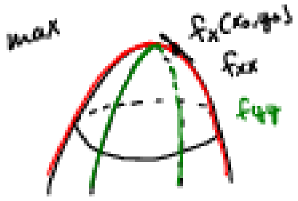
$$\underline{D(a,b)} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

a) If $D > 0$ + $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min

b) If $D > 0$ + $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max

c) If $D < 0$ then (a,b) is a saddle point.

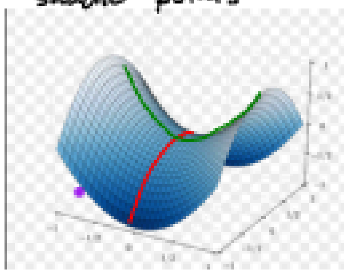
d) If $D = 0$ inconclusive.



mins

c s

saddle points



$A_{ij} = [f_{x_i x_j}(\vec{a})]$

Det (A)

$\rightsquigarrow$ "Curvature" Do Carmo Hessian matrix

Example: Find the shortest distance from $\overset{P_0}{(1,0,-2)}$ to the plane $\underline{x+2y+z=4}$

$D(a,b) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a,b)\,f_{yy}(a,b) - [f_{xy}(a,b)]^2$ ← Hessian matrix

a) If $D>0$ & $f_{xx}(a,b)>0$, then $f(a,b)$ is a local min

b) If $D>0$ & $f_{xx}(a,b)<0$, then $f(a,b)$ is a local max

c) If $D<0$ then $f(a,b)$ is a saddle point.

d) If $D=0$ inconclusive.

$$d = \sqrt{(x-1)^2+(y)^2+(z+2)^2} = d(P, P_0) \qquad \underline{P=(x,y,z)}$$

$$d^2 = (x-1)^2+(y)^2+(z+2)^2$$

$$z = 4-x-2y$$

$$f(x,y) = d^2(x,y) = (x-1)^2 + y^2 + (-x-2y+6)^2 \longleftarrow \text{Find min of this}$$

$$= x^2-2x+1+y^2+x^2+4y^2+4xy-12x-12y+36$$

$$= 2x^2+5y^2+4xy-14x-12y+37$$

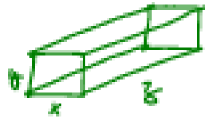
$f_x = 4x+4y-14 \qquad f_{xx} = 4 \qquad f_{xy} = 4 \qquad D = 40-16 > 0$

$f_y = 10y+4x-12 \qquad f_{yy} = 10$

$f_x = 4x+4y-14 = 0$

$f_y = 10y+4x-12 = 0$

Example: Find the max volume of a box to be made from $12\,cm^2$ of cardboard.



$$D(a,b) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a,b)\,f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Hessian matrix

a) If $D>0$ & $f_{xx}(a,b)>0$, then $f(a,b)$ is a local min

b) If $D>0$ & $f_{xx}(a,b)<0$, then $f(a,b)$ is a local max

c) If $D<0$ then (a,b) is a saddle point.

d) If $D=0$ inconclusive.

$$V(x,y,z) = xyz$$

Express z in terms of x & y

$$SA = 2xy + 2yz + 2xz = 12$$

$$z\,2(y+x) = 12 - 2xy$$

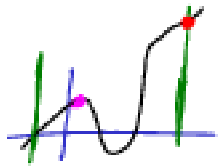
$$z = \frac{12-2xy}{2(y+x)}$$

y u

$$\boxed{V(x,y) = \frac{xy(12-2xy)}{2(y+x)}}$$

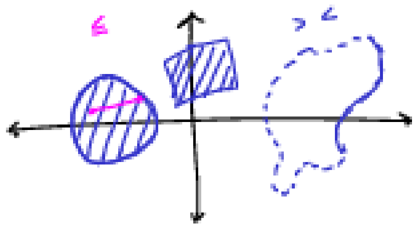
Apply 2nd derivative test to.

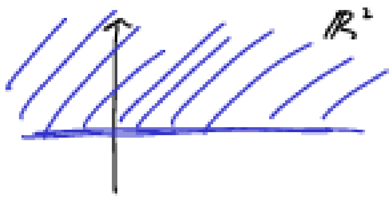
□ Absolute max/min values.



A little Topology.

Closed, open, & clopen sets. Bounded sets.





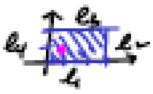
$(x,y): x \geq 0$

Theorem: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$ + an absolute minimum value $f(x_2,y_2)$.

1. Find the critical points in D
2. Find extreme values on D
3. Compare

Example: Find the absolute max/min values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle

$D = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le 2\}$



1. $f_x = 2x - 2y = 0 \Rightarrow 2x = 2y \Rightarrow x = y$
   $f_y = -2x + 2 = 0 \Rightarrow 2 = 2x \Rightarrow x = 1$ $\Rightarrow$ critical point is $(1,1)$

2. $f_{xx} = 2$ $\quad f_{xy} = -2$
   $f_{yy} = 0$ $\quad f_{yx} = -2$

   $D = 0 - (-2)^2 = -4 < 0 \Rightarrow$ $(1,1)$ is a saddle point

3. Parametrize the boundary.

   $\ell_1(x) = \langle x, 0\rangle : 0 \le x \le 3$

   $f(\ell_1) = x^2 \rightsquigarrow$ min along ℓ_1 is at $(0,0)$ min : 0
   max along ℓ_1 is at $(3,0)$ max : 9

   $f(\ell_2)$ $\ell_2(y) = \langle 3, y\rangle : 0 \le y \le 2$

   $f(\ell_2(y)) = 9 - 6y + 2y = 9 - 4y$ min along ℓ_2 is at $(3,2)$: 1
   max along ℓ_2 is at $(3,0)$: 9

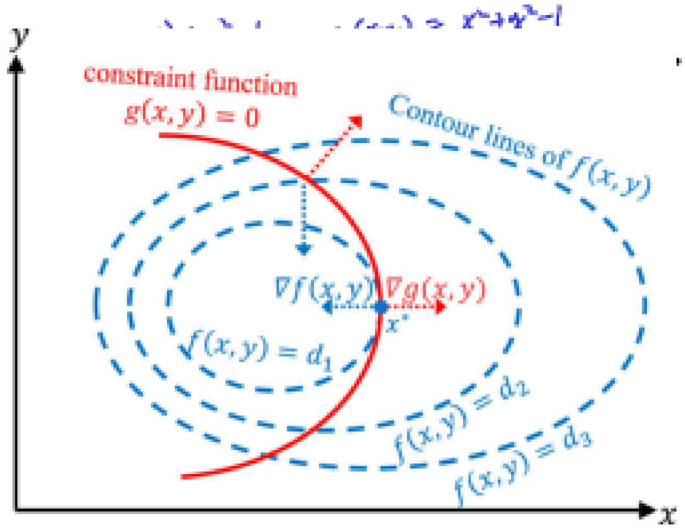
$D = \{(x,y) : x^2 + y^2 = 1\}$

$\vec{r}(\theta) = \langle \cos\theta, \sin\theta\rangle$

$f(\theta) = \cos^2\theta - 2\cos\theta\sin\theta + 2\sin\theta$

Section 14.8: The method of Lagrange Multipliers

$g(x,y) = x^2+y^2-1$



Theorem.

- To find the max/min values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ [assuming these values exist & $\nabla f \ne \vec{0}$ on $g(x,y,z) = k$]:

  a) Find all values x, y, z, and λ such that

  $\boxed{\nabla f(x,y,z) = \lambda \nabla g(x,y,z)}$

  and $g(x,y,z) = k$

  b) Evaluate f at all points (x,y,z) satisfying a). Compare.

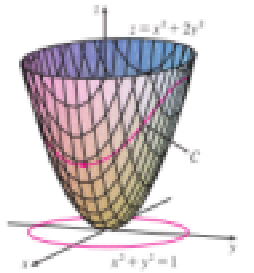


FIGURE 2

The geometry behind the use of Lagrange multipliers in Example 2 is shown in Figure 3. The extreme values of $f(x,y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$.

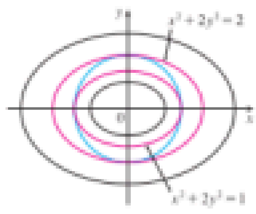


FIGURE 3

$x = \cos t$
$y = \sin t$

Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

$g(x,y) = x^2 + y^2$

$\nabla f = \langle 2x, 4y\rangle$ $\quad \nabla g = \langle 2x, 2y\rangle$

$\nabla f = \lambda \nabla g$

$$\boxed{\begin{aligned} 2x &= \lambda 2x \\ 4y &= \lambda 2y \\ x^2 + y^2 &= 1 \end{aligned}}$$

*Be Careful! You can't divide by zero!

when x is not zero, $\lambda = 1$
$4y = 2y \Rightarrow y = 0$ & $x = \pm 1$
when $x = 0$, $y^2 = 1 \Rightarrow y = \pm 1$

$\boxed{(\pm 1, 0) \quad (0, \pm 1)}$

$f(\pm 1, 0) = 1 \longleftarrow$ min.
$f(0, \pm 1) = 2 \longleftarrow$ max

$f(t) = \cos^2 t + 2\sin^2 t$

$\ell(x,y,\lambda) = x^2 + 2y^2 + \lambda x^2 + \lambda y^2 - \lambda$

$\ell_x = 2x + 2\lambda x = 0$
$\ell_y =$
$\ell_\lambda =$

Example: Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$.

□ Two Constraints

$$\nabla f = \sum_{i=1}^{n} \lambda_i \nabla g_i \quad \longleftrightarrow \quad \nabla f = \int \lambda(x)\, g$$

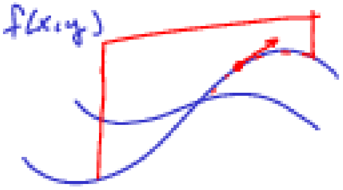
$$\boxed{\begin{aligned} \nabla f(x_0,y_0,z_0) &= \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0) \\ g(x_0,y_0,z_0) &= k_1 \\ h(x_0,y_0,z_0) &= k_2 \end{aligned}}$$

Example: Find the max/min of $f(x,y,z) = x+2y+3z$ on the curve of intersection of the plane $x-y+z=1$ and the cylinder $x^2+y^2=1$.

Yesterday: Directional derivative, gradient vector.

f_x $\quad$ f_y

$f(x,y)$



direction = unit vector

$\vec{u} = \langle a, b \rangle$

$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b}$$ Theorem

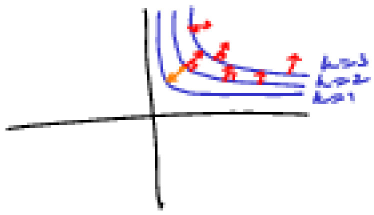
nabla

$\nabla f(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\boxed{\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle}$$

$\nabla f(x_1, \dots, x_n) = \langle f_{x_1}(x_1, \dots, x_n), f_{x_2}(x_1 \dots), \dots$

$f_{x_n}(\sim) \rangle$



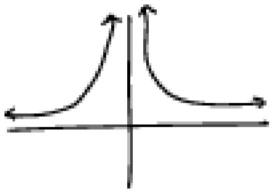
$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}}$$

$|\nabla f(x, y)| \, |\vec{u}| \cdot \cos\theta$

1

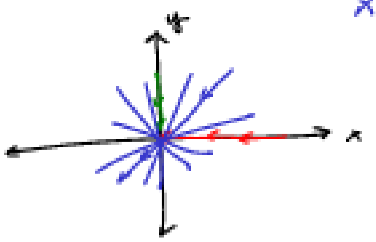
limits

$$f(x,y) = \frac{3xy^3}{x^2 + y^6}$$



$$\lim_{(x,y)\to(0,0)} \frac{x^{500} - y^{500}}{x^2 + y^2}$$

$\frac{x^2}{x} \rightsquigarrow x$



$\lim_{(x,y)\to(0,0)} f(x,y)$

$$\boxed{\text{Plug in } (0,0), \longrightarrow \frac{0}{0} \longrightarrow \text{DNE}}$$

$$\lim_{x\to 0} \frac{0}{0 + y^6} = 0$$

$y = mx$

$$\lim_{x\to 0} \frac{3 \cdot x \cdot m^3 x^3}{x^2 + m^6 x^6} = \lim_{x\to 0} \frac{3m^3 x^4}{x^2 + m^6 x^6} = \lim_{x\to 0} \frac{3m^3 x^2}{1 + m^6 x^4} = 0$$

Let $x = y^3$

$$\lim_{y\to 0} \frac{3\cancel{y^6}}{2\cancel{y^6}} = \lim_{y\to 0} \frac{3}{2} = \frac{3}{2} \neq 0 \Rightarrow \text{DNE}$$

r^{500}

$x = r\cos\theta \quad y = r\sin\theta$

$$\left| \lim_{(x,y)\to(0,0)} \frac{x^{500} - y^{500}}{x^2 + y^2} \right| \le 0 \quad \lim_{r\to 0} \frac{r^{500}(\cos^{500}\theta - \sin^{500}\theta)}{r^2} = 0$$

Participation Points Questions

1. Let $f(x,y) = xy + y^2 - x^3$. The point $(0,0)$ is a critical point of f. Is $(0,0)$ a local min, max, or a saddle point?

2. Calculate ∇f, where $f(x, \ ) = xy + y^2 - x^3$.