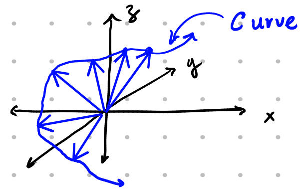
Thursday, June 30, 2022

MATH 164 Lecture Notes

Chapter 13 : Vector Functions

Section 13.1: Vector Functions and Space Curves

$\vec{r}(t) = \langle t^2, \sin(t), 6t \rangle$



- Definition + component functions

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

- Limits and Continuity

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle$$

A vector function is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

Example: Find the limit

$$\lim_{t\to 0} \left(\vec{r}(t) = \left\langle 1+t^3, te^{-t}, \frac{\sin(t)}{t} \right\rangle \right)$$

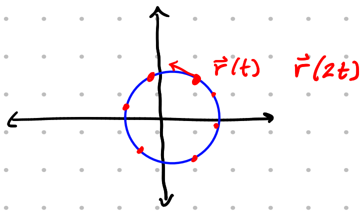
$$= \left\langle \lim_{t\to 0} 1+t^3, \lim_{t\to 0} te^{-t}, \lim_{t\to 0} \frac{\sin(t)}{t} \right\rangle$$

$$= \langle 1, 0, 1 \rangle$$

- Space Curves

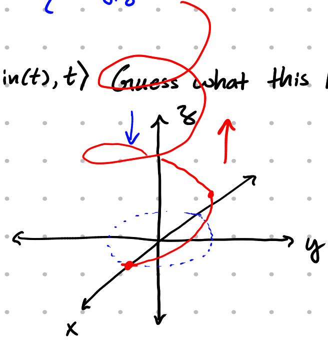
A space curve is the trace of a vector equation $\vec{r}(t)$. That is,

$$C = \{ (x,y,z) \in \mathbb{R}^3 : x = x(t), y = y(t), z = z(t) \}$$

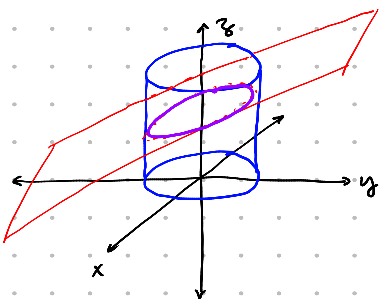


Example: $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ Guess what this looks like!

- * projections
- * limiting behavior $t \to \pm\infty$?



Example: Find a vector function representing the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.



$\vec{r}(t)$

Any point (x,y,z) in the intersection must satisfy

$$x^2+y^2=1$$
and
$$y+z=2.$$

* Parameterize the cylinder:

$x = \cos(t)$

$y = \sin(t)$

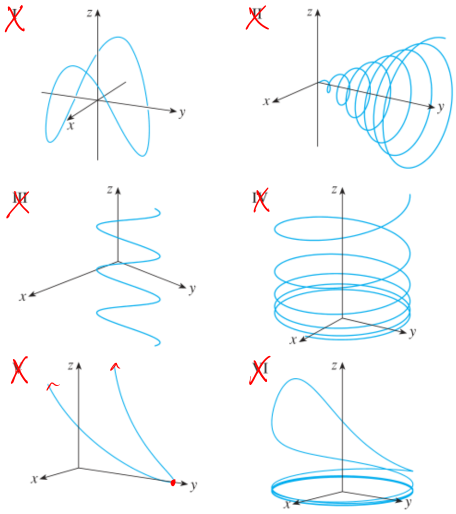
$z = z$

$$\boxed{\vec{r}(t) = \langle \cos(t), \sin(t), 2-\sin(t) \rangle}$$

$z = 2 - y = 2 - \sin(t)$

Example:

21–26 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.



21. $x = t\cos t$, $y = t$, $z = t\sin t$, $t \geqslant 0$ II

22. $x = \cos t$, $y = \sin t$, $z = 1/(1 + t^2)$ VI $\lim_{t\to\infty} z(t) = 0$

23. $x = t$, $y = 1/(1 + t^2)$, $z = t^2$ V

24. $x = \cos t$, $y = \sin t$, $z = \cos 2t$ I

25. $x = \cos 8t$, $y = \sin 8t$, $z = e^{0.8t}$, $t \geqslant 0$ IV

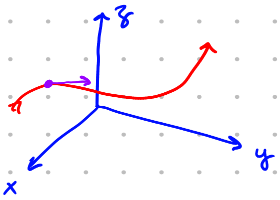
26. $x = \cos^2 t$, $y = \sin^2 t$, $z = t$ III

$z = 0 \Rightarrow x = 0 \Rightarrow t = 0 \Rightarrow y = 1$

Section 13.2: Derivatives and Integrals of Vector Functions

* If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ where f, g, and h are differentiable functions, then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$ "velocity"



- Unit tangent vectors + tangent lines

$$\boxed{\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$$

Example: Find parametric equations for the tangent line to the helix $\vec{r}(t) = \langle 2\cos(t), \sin(t), t \rangle$ at the point $(0, 1, \pi/2)$.

$\vec{r}_0 + \vec{v}t$ $\qquad 2\cos(t) = 0, \quad \sin(t) = 1, \quad t = \pi/2$

* A point + a vector

$$\vec{r}\,'(t) = \langle -2\sin(t), \cos(t), 1 \rangle$$

$$\vec{r}\,'(\pi/2) = \langle -2, 0, 1 \rangle = \vec{v}$$

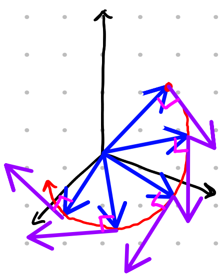
$$\vec{\ell}(t) = \langle 0, 1, \pi/2 \rangle + \langle -2, 0, 1 \rangle t$$

$$\boxed{\begin{aligned} x(t) &= -2t \\ y(t) &= 1 \\ z(t) &= \pi/2 + t \end{aligned}}$$

Differentiation Rules

1. $\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$
2. $\frac{d}{dt}\left(c\,\vec{u}(t)\right) = c\frac{d}{dt}\vec{u}(t)$
3. $\frac{d}{dt}\left(f(t)\,\vec{u}(t)\right) = \left(\frac{d}{dt}f(t)\right)\vec{u}(t) + f(t)\left(\frac{d}{dt}\vec{u}(t)\right)$ → $f(t)\vec{v}(t)$ still a vector function
4. $\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \left(\frac{d}{dt}\vec{u}(t)\right)\cdot\vec{v}(t) + \vec{u}(t)\cdot\left(\frac{d}{dt}\vec{v}(t)\right)$ → $\vec{v}(t)\cdot\vec{u}(t) : \mathbb{R} \to \mathbb{R}$
5. $\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \left(\frac{d}{dt}\vec{u}(t)\right)\times\vec{v}(t) + \vec{u}(t)\times\left(\frac{d}{dt}\vec{v}(t)\right)$ → $\vec{v}(t)\times\vec{u}(t)$ still a vector function
6. $\frac{d}{dt}\left(\vec{u}(f(t))\right) = f'(t)\,\vec{u}\,'(f(t))$ * Chain Rule.

Example: Show that if $|\vec{r}(t)| = c$ then $\vec{r}\,'(t) \perp \vec{r}(t)\ \forall t$.



$$\vec{v}\cdot\vec{v} = |\vec{v}|^2 \Rightarrow \vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$$

$$0 = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 2\,\vec{r}\,'(t)\cdot\vec{r}(t)$$

$$\Rightarrow \vec{r}\,'(t)\cdot\vec{r}(t) = 0 \quad \therefore \vec{r}\,'(t) \perp \vec{r}(t) \qquad \square$$

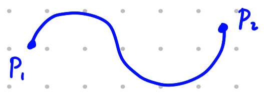
- Integrals

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t), \int_a^b g(t), \int_a^b h(t) \right\rangle$$

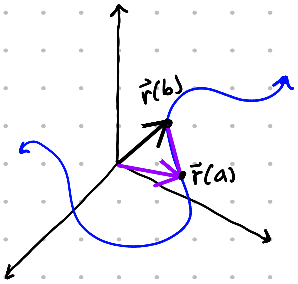
Section 13.3 Arc Length and Curvature

(intrinsic properties) Don't depend on parameterization



Arc Length: If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$



$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\, dt}$$

Example: Find the length of the arc of the helix with vector equation

$$\vec{r}(t) = \langle -\sin(t), \cos(t), t \rangle$$

from $(1, 0, 0)$ to $(1, 0, 2\pi)$

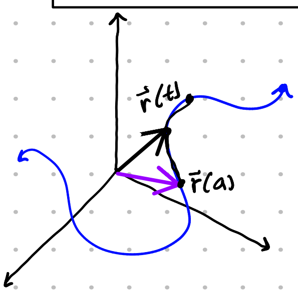
$$L = \int_0^{2\pi} \sqrt{\cos^2(t) + \sin^2(t) + 1}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

▫ Arc Length Function.

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\, du}$$

take the derivative of both sides with respect to t,
FTC

$$\Rightarrow \quad \frac{ds}{dt} = |\vec{r}\,'(t)|$$



* Reparameterize a vector function by arc length.

Example: Reparameterize the helix $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ by arc length starting at $(1,0,0)$

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{2}$$

$$s(t) = \int_0^t |\vec{r}\,'(u)|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t$$

$$\vec{r}(s(t)) = \langle \cos(\sqrt{2}\, t), \sin(\sqrt{2}\, t), \sqrt{2}\, t \rangle$$

$$t = \frac{s}{\sqrt{2}}$$

$$\boxed{\vec{r}(t(s)) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle} \quad ✓$$

$\vec{r}\,''(s)$ = curvature at $\vec{r}(s)$

□ Curvature

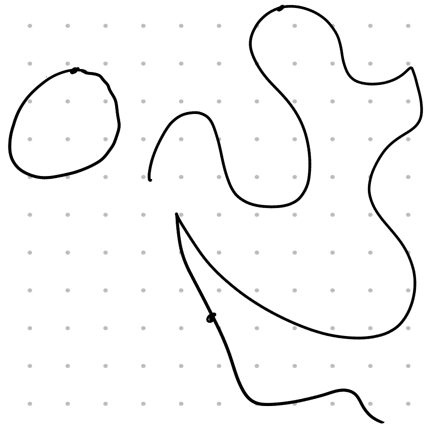
Definition: The curvature of a curve is

$$\boxed{\kappa = \left| \frac{d\vec{T}}{ds} \right|}$$

where $\vec{T}$ is the unit tangent vector.

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} \qquad \kappa = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}/dt}{ds/dt} \right| \qquad ds/dt = |\vec{r}\,'(t)|$$

$$\boxed{\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}}$$



Example: Find the curvature of a circle of radius a.

$\vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle$

$\vec{r}\,'(t) = \langle -a\sin(t), a\cos(t) \rangle$

$|\vec{r}\,'(t)| = a \qquad \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin(t), \cos(t) \rangle$

$\vec{T}'(t) = \langle -\cos(t), -\sin(t) \rangle$

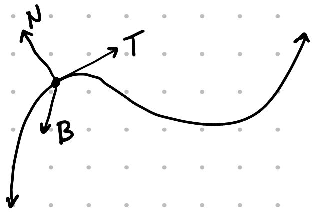
$$\boxed{\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{a}}$$

Theorem: The curvature of a curve given by $\vec{r}$ is

$$\boxed{\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

Example: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

□ The Normal + Binormal Vectors.

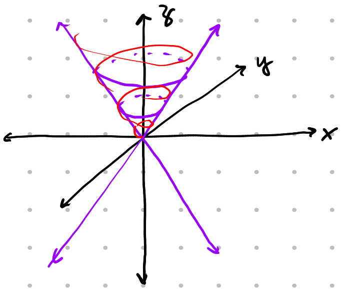


Normal: $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$

BiNormal: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

More problems from 13.1

27) Show that the curve $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2+y^2$.



$$t^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\underbrace{\cos^2 t + \sin^2 t}_{1})$$

$$t^2 = t^2 \checkmark$$

43) Find a vector function representing the curve of intersection $z = \sqrt{x^2+y^2}$ & the plane $z = 1+y$

$$\sqrt{x^2+y^2} = 1+y$$

$$x^2 + \cancel{y^2} = (1+y)^2 = 1+2y+\cancel{y^2}$$

$$x^2 = 1+2y \rightsquigarrow y = \tfrac{1}{2}(x^2-1)$$

$$\boxed{\vec{r}(t) = \left\langle t, \tfrac{1}{2}(t^2-1), 1+\tfrac{1}{2}(t^2-1)\right\rangle}$$

Two main strategies

1. Parameterize one surface directly (circles ellipses)
2. Set thing equal to each other & pick one variable to be the parameter.

46) Semiellipsoid $x^2+y^2+4z^2=4$, $y \geq 0$
And the cylinder $x^2+z^2=1$

Let $x = \cos t$, $z = \sin t$

$$\cos^2 t + y^2 + 4\sin^2 t = 4$$

$$y^2 = 4 - \cos^2 t - 4\sin^2 t$$

$$y = \sqrt{4-\cos^2 t - 4\sin^2 t}$$

$$\vec{r}(t) = \left\langle \cos t, \sqrt{4-\cos^2 t - 4\sin^2 t}, \sin(t)\right\rangle$$

13.2: If $\vec{r}_1$ & $\vec{r}_2$ are two vector functions & the intersect at time t_0, I could ask you to find the angle between the curves at t_0.

1. Calculate $\vec{r}_1{}'(t_0)$ & $\vec{r}_2{}'(t_0)$
2. Calculate $\vec{r}_1{}'(t_0) \cdot \vec{r}_2{}'(t_0)$ to find θ

Announcements:

- 1st exam is a week from Monday. (12.1-14.2)
- Watch for an email about scheduling
- Do your Webwork!

Participation Points Questions

1. Find the limit

$$\lim_{t \to 4} \vec{r}(t) \quad \text{where } \vec{r}(t) = \left\langle t^2, \frac{t^2}{2-t^2}, t+2 \right\rangle$$

2. What is the curvature of the circle given by

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle \quad ?$$