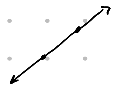
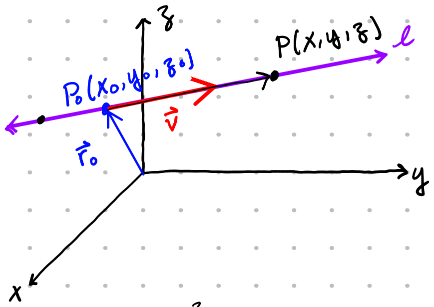
Wednesday, June 29, 2022
MATH 164 Lecture Notes

Section 12.5: Equations of Lines and Planes



▫ Lines



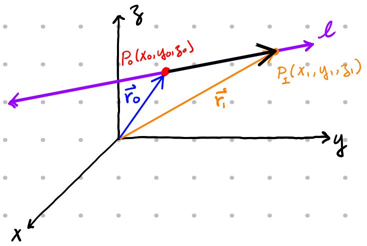
- What is the minimum "data" we need to define a line?

The vector equation for a line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$\overrightarrow{P_0P} = t\vec{v}$

The line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1$$



$$\overrightarrow{P_0P_1} = \langle x_1 - x_0,\, y_1 - y_0,\, z_1 - z_0\rangle$$

$$\vec{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle x_1 - x_0,\, y_1 - y_0,\, z_1 - z_0\rangle$$

$$\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

$\boxed{t : \text{parameter}}$

$$\vec{r}(t) = \vec{r}_0 + t\vec{r}_1 - t\vec{r}_0$$
$$= (1-t)\vec{r}_0 + t\vec{r}_1$$

$\langle x,y,z\rangle \rightsquigarrow$ position of a particle
$t \rightsquigarrow$ time.

The parametric equations for a line in $\mathbb{R}^3$:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

direction vector $\vec{v}$

$$\langle x(t), y(t), z(t)\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$
$$= \langle x_0 + at,\, y_0 + bt,\, z_0 + ct\rangle$$

$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
$$z(t) = z_0 + ct$$

Example: Find the parametric equations for the line that passes through the point $(5,1,3)$ & is parallel to the vector $\langle 1, 4, -2\rangle$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$(5,1,3) \rightsquigarrow \langle 5,1,3\rangle \rightsquigarrow x_0 = 5,\ y_0 = 1,\ z_0 = 3$
$\langle a,b,c\rangle \rightsquigarrow \langle 1,4,-2\rangle \rightsquigarrow a = 1,\ b = 4,\ c = -2$

$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
$$z(t) = z_0 + ct$$

$$\boxed{\begin{aligned} x(t) &= 5 + t \\ y(t) &= 1 + 4t \\ z(t) &= 3 - 2t \end{aligned}}$$

vector equation:

$$\langle x(t), y(t), z(t)\rangle = \langle 5,1,3\rangle + t\langle 1,4,-2\rangle$$

The symmetric equations for a line in $\mathbb{R}^3$:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

when $a,b,c \neq 0$. If $a=0$, we have:

$$x-x_0=0 \quad \& \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$x(t) = x_0 + at$
$y(t) = y_0 + bt$
$z(t) = z_0 + ct$

$$\boxed{\begin{aligned} x-x_0 &= 0 \\ y-y_0 &= 0 \\ \frac{z-z_0}{c} &= t \end{aligned}}$$

Solve for t

$\frac{x-x_0}{a} = t$

$\frac{y-y_0}{b} = t$

$\frac{z-z_0}{c} = t$

Example: Show that the lines L_1 and L_2 are skew.

Skew means not parallel & also not intersecting

$L_1: x = 1+t \quad y = -2+3t \quad z = 4-t$

$L_2: x = 2s \quad y = 3+s \quad z = -3+4s$

1. Not parallel:

$\vec{v}_1 = \langle 1, 3, -1 \rangle$
$\vec{v}_2 = \langle 2, 1, 4 \rangle$

$\vec{v}_1 = k \cdot \vec{v}_2$
$1 = k \cdot 2 \leadsto k = \frac{1}{2}$
$3 = k \cdot 1 \quad 3 \neq \frac{1}{2}$ ✓
$-1 = k \cdot 4$

2. Show that L_1 & L_2 do not intersect.

If L_1 & L_2 intersect, then there exists some point (x,y,z) such that:

$1+t = 2s \leadsto t = 2s-1$
$-2+3t = 3+s \qquad -2+(6s-3) = 3+s$
$4-t = -3+4s \qquad 6s-5 = 3+s$
$5s = 8 \Rightarrow s = 8/5$
$t = \frac{16}{5} - \frac{5}{5} = \frac{11}{5}$

$7 = \frac{11}{5} + \frac{32}{5} = \frac{43}{5}$ not true $\Rightarrow$ L_1 & L_2 do not intersect!

Example: Find the parametric equations and symmetric equations for the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$. At what point does the line pass through the xy-plane?

$x(t) = x_0 + at$
$y(t) = y_0 + bt$
$z(t) = z_0 + ct$

$\vec{v} = \overrightarrow{AB} = \langle \overset{a}{1}, \overset{b}{-5}, \overset{c}{4} \rangle$
$\langle x_0, y_0, z_0 \rangle = \langle 2, 4, -3 \rangle$

parametric

$$\boxed{\begin{aligned} x(t) &= 2+t \\ y(t) &= 4-5t \\ z(t) &= -3+4t \end{aligned}} \leadsto \boxed{x-2 = \frac{-y+4}{5} = \frac{z+3}{4}}$$

symmetric

To find the point where the line passes through the xy-plane

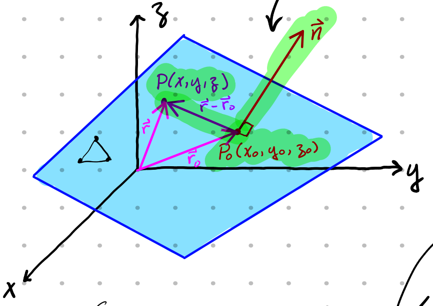
when $z(t) = 0$, $0 = -3+4t \leadsto$ Solve for t: $4t = 3$ $\boxed{t = \frac{3}{4}}$

$x(3/4) = 2 + \frac{3}{4} = \frac{11}{4}$
$y(3/4) = 4 - \frac{5 \cdot 3}{4} = \frac{1}{4}$
$z(3/4) = 0$

$$\boxed{\left(\frac{11}{4}, \frac{1}{4}, 0\right)}$$

☐ Planes

Normal vector



$\ell = \{(x,y) \in \mathbb{R} : \langle a,b\rangle \cdot \langle x-x_0, y-y_0\rangle = 0\}$

$y = \frac{-a}{b}x + c$

"linear equation"

- What is the minimum "data" we need to define a plane?

  * 3 points * 1 point + a vector

The vector equation for a plane: $\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$

$$\langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

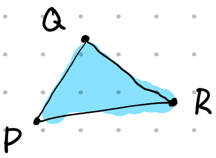
$$= ax+by+cz+\underbrace{\left(-(ax_0+by_0+cz_0)\right)}_{d} = 0$$

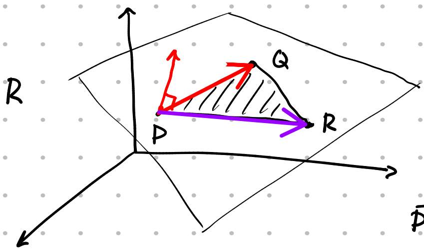
The scalar equation for the plane that passes through $P_0(x_0,y_0,z_0)$ and with normal vector $\vec{n} = \langle a,b,c\rangle$: $$ax+by+cz+d=0$$ where $d = -(ax_0+by_0+cz_0)$

Plane $= \{(x,y,z) : ax+by+cz+d = 0\}$

$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$

Example: Find the plane passing through $P(1,3,2)$ $Q(3,-1,6)$ and $R(5,2,0)$





A vector lying in the plane? $\vec{PQ}$, $\vec{PR}$

$\vec{n} = \vec{PQ} \times \vec{PR}$

$\vec{PQ} = \langle 2,-4,4\rangle$ $\vec{PR} = \langle 4,-1,-2\rangle$

$$\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 8+4, 4+16, -2+16\rangle = \langle 12, 20, 14\rangle$$

$\vec{n} = \langle 6, 10, 7\rangle$

$\langle 6,10,7\rangle \cdot \langle x-5, y-2, z\rangle$

$= 6x-30+10y-20+7z = 0$

$= \boxed{6x+10y+7z-50=0}$

Example: Find the point where $\vec{r}(t) = \langle \overset{x(t)}{2+3t}, \overset{y(t)}{-4t}, \overset{z(t)}{5+t}\rangle$ intersects $4x+5y-2z=18$

$x(t) = 2+3t$
$y(t) = -4t$
$z(t) = 5+t$

$\exists\ (x,y,z)$ satisfying

$4x+5y-2z = 18$
$4(2+3t)+5(-4t)-2(5+t) = 18$

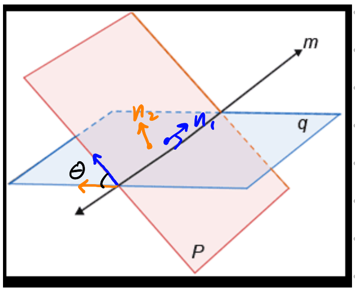
$$\begin{array}{r} 22 \\ 12 \\ \hline 10 \end{array}$$

$8+12t-20t-10-2t = -10t-2 = 18 \rightsquigarrow -10t = 20 \rightsquigarrow \boxed{t=-2}$

$x(-2) = 2-6 = -4$
$y(-2) = -4(-2) = 8$
$z(-2) = 5-2 = 3$

$\boxed{(-4,8,3)}$

Example: Find the angle between $\overset{P_1}{x+y+z=1}$ and $\overset{P_2}{x-2y+3z=1}$.
Find an equation for their line of intersection.



• Find θ.

$\vec{n}_1 = \langle 1,1,1\rangle$
$\vec{n}_2 = \langle 1,-2,3\rangle$

$\cos(\theta) = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

$\sin(\theta) = \dfrac{|\vec{n}_1 \times \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

• Find line of intersection. (point + a vector)

$\vec{r}(t) = \vec{r}_0 + \vec{v}t \qquad \vec{v} = \vec{n}_1 \times \vec{n}_2$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5,-2,-3\rangle$$

$x+y+z=1$
$x-2y+3z=1$

Let $z=0$

$x+y=1$
$-(x-2y=1)$
$3y=0 \Rightarrow y=0$

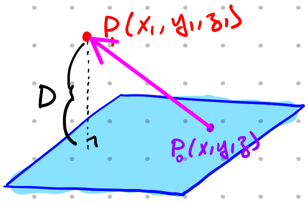
$(1,0,0)$ satisfies both plane equations simultaneously.

$\vec{r}_0 = \langle 1,0,0\rangle$

$$\boxed{\vec{r}(t) = \langle 1+5t, -2t, -3t\rangle}$$

▫ Distances

Example: Find the distance between $P_1(x_1,y_1,z_1)$ and $ax+by+cz+d=0$



$\overrightarrow{P_0P_1} = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$

Project $\overrightarrow{P_0P_1}$ onto $\vec{n} = \langle a,b,c\rangle$

$$D = \left|\operatorname{comp}_{\langle a,b,c\rangle} \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle\right|$$

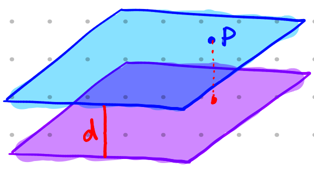
$$= \left|\frac{\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle \cdot \langle a,b,c\rangle}{|\langle a,b,c\rangle|}\right| = \left|\frac{a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)}{\sqrt{a^2+b^2+c^2}}\right|$$

$$= \left|\frac{ax_1+by_1+cz_1-(ax_0+by_0+cz_0)}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right| = \text{distance}$$

Example: Find the distance between $10x+2y-2z=5$ and $5x+y-z=1$

$\vec{n}_1 = \langle 10,2,-2\rangle \qquad \vec{n}_1 = 2\vec{n}_2 \quad \therefore 10x+2y-2z=5$ is parallel $5x+y-z=1$

$\vec{n}_2 = \langle 5,1,-1\rangle$



1. Find a point on one plane, called $P_1(x_1,y_1,z_1)$
2. Calculate distance from P_1 to the other plane using the previous example.

Example: Find the distance between the lines

$$L_1: x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

$$L_2: x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

1. Find two planes Plane 1 + Plane 2 such that L_1 is in Plane 1, L_2 is in Plane 2, + Plane 1 is parallel to Plane 2.

2. Calculate the distance between Plane 1 + Plane 2

Plane 1 + Plane 2 have the same normal vector:

$$\left.\begin{array}{l} \vec{n} \perp \vec{v}_1 \\ \vec{n} \perp \vec{v}_2 \end{array}\right\} \Rightarrow \vec{n} = \vec{v}_1 \times \vec{v}_2 \qquad \begin{array}{l} \vec{r}_1 = \langle 1, -2, 4 \rangle \\ \vec{r}_2 = \langle 0, 3, -3 \rangle \end{array}$$

Plane 1: $\vec{n} \cdot \langle x-1, y+2, z-4 \rangle = 0$

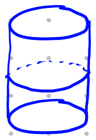
Plane 2: $\vec{n} \cdot \langle x, y-3, z+3 \rangle = 0$

Then find distance between Plane 1 + Plane 2.

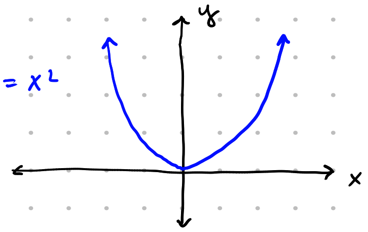
Break: X : 10

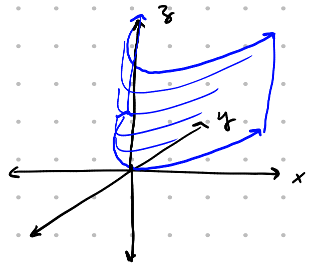
Section 12.6: Cylinders and Quadric Surfaces

- Cylinders



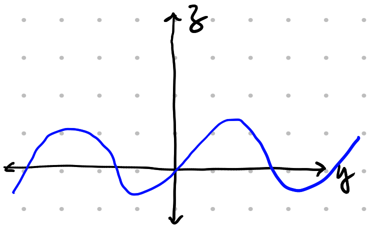
$y = x^2$

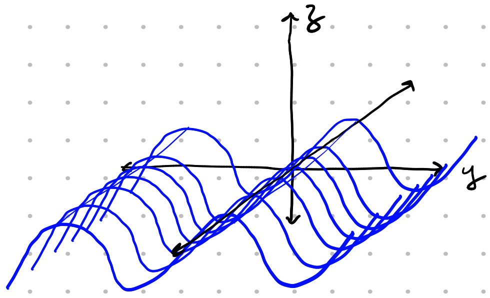




Any surface formed by "dragging" a curve in 2-d is called a cylinder.

$z = \sin(y)$





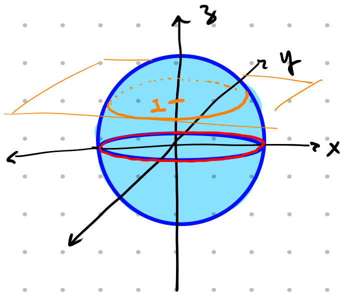
- Quadric Surfaces

Plane: $ax + by + cz + d = 0$

  - $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

  - Traces

Sphere: $4 = x^2 + y^2 + z^2$



A trace is a curve you get from setting 1 variable equal to a constant

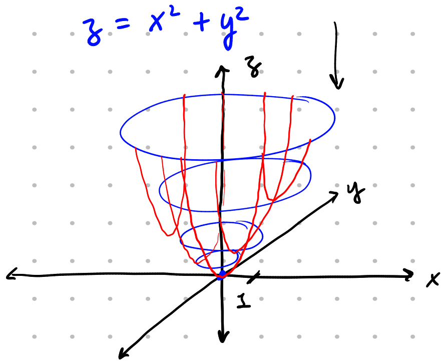
Find the trace when $z = 0$

$4 = x^2 + y^2 \rightsquigarrow$ equation for a circle in the xy-plane

Find the trace when $z = 1$

$4 = x^2 + y^2 + 1 \rightsquigarrow 3 = x^2 + y^2$

$z = x^2 + y^2$



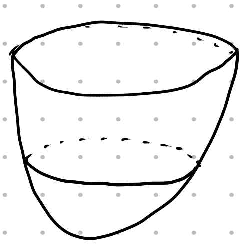
Traces for $z = k$

$k = x^2 + y^2$

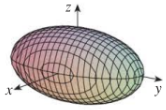
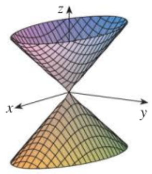
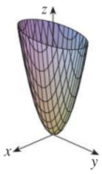
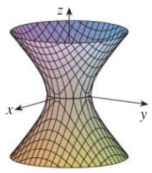
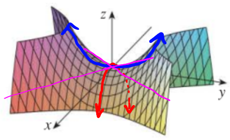
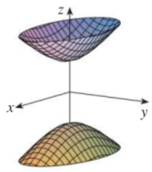
Traces for $x = k$

$z = k^2 + y^2$

$z = y^2$



Paraboloid

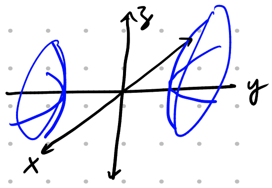
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
$\boxed{\text{Hyperbolic Paraboloid}}$ 	If $x=0$ $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ $z=-y^2$ $y=0$ Horizontal traces are hyperbolas. $z=0$ $x^2=y^2$ Vertical traces are parabolas. The case where $c<0$ is illustrated. $x=\pm y$	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Example: Classify the surface described by $4x^2-y^2+2z^2+4=0$

$4x^2-y^2+2z^2=-4$

$-4x^2+y^2-2z^2=4$

$-x^2+\frac{y^2}{4}-\frac{z^2}{2}=1$



Participation Points Problems

1. Are the lines described below parallel or not? Why?

$$L_1(t) = \langle 1+6t, 8t, 2+4t \rangle$$
$$L_2(s) = \langle 3t, 1+4t, 5+2t \rangle$$

2. What type of quadric surface is $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$?