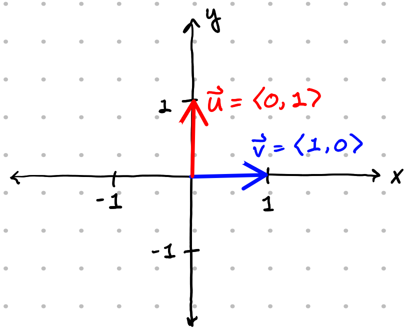
# Tuesday, June 28, 2022

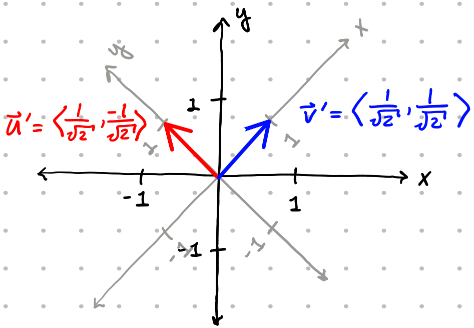
## MATH 164 Lecture Notes

### Section 12.3: Dot Products

* Why can't we just multiply vectors component-wise to get new vectors?




$\langle 0, 1 \rangle \star \langle 1, 0 \rangle = \langle 0, 0 \rangle$




$\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle \star \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \langle \frac{1}{4}, \frac{-1}{4} \rangle$

**Definition:** If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the dot product of $\vec{a}$ and $\vec{b}$ is the <u>number</u> $\vec{a} \cdot \vec{b}$ given by

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

In n dimensions: $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$

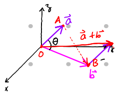
$\vec{a} = \langle 1, 2, 3 \rangle \quad \vec{b} = \langle 1, 0, 5 \rangle$

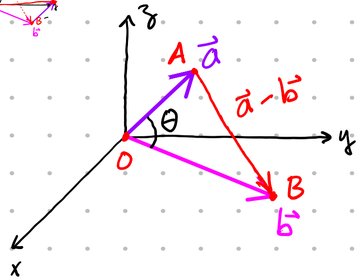
$$\vec{a} \cdot \vec{b} = 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 5$$
$$= 1 + 0 + 15 = 16$$

**Properties of the Dot Product:**

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
3. $\vec{0} \cdot \vec{a} = \vec{0}$
4. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
5. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$

**Theorem:** If $\theta$ is the angle between $\vec{a}$ and $\vec{b}$ then $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$ (Geometric definition)






**Proof:**

1: Law of Cosines: $|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$

2: Convert to vectors: $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

3. $(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$$= |\vec{a}|^2 - 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

4. $\cancel{|\vec{a}|^2} - 2\,\vec{a}\cdot\vec{b} + \cancel{|\vec{b}|^2} = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}||\vec{b}|\cos\theta$

$$\cancel{2}\,\vec{a}\cdot\vec{b} = \cancel{2}\,|\vec{a}||\vec{b}|\cos\theta \qquad \square$$

**Corollary:** If $\theta$ is the angle between $\vec{a}$ & $\vec{b}$, then $\boxed{\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}}$

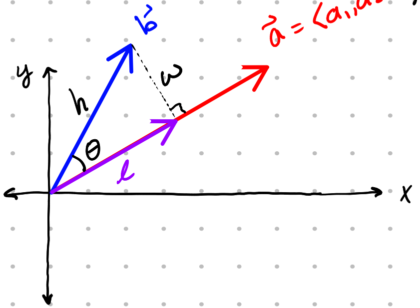
In particular, $\vec{a} \perp \vec{b} \iff \vec{a}\cdot\vec{b} = 0$

## ☐ Projections

∘ Scalar Projection

### ∘ Vector Projection

$\vec{a} = \langle a_1, a_2 \rangle$ Projection of $\vec{b}$ onto $\vec{a}$




$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\cdot\left(\frac{\vec{a}}{|\vec{a}|}\right) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

Find a vector which is a scalar multiple of $\vec{a}$ with length $\ell$:

$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\ell}{h} \Rightarrow \frac{\cos(\theta)}{|\vec{b}|} = \ell$$

abuse of notation.

### ∘ Scalar Projection

$$\text{comp}_{\vec{a}}\vec{b} = |\text{proj}_{\vec{a}}\vec{b}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}|\vec{a}| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

Scalar projection is a scalar

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}}$$

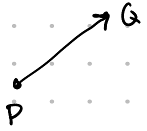
vector projection is a vector.

## Work and Force

The work done by a constant force $\vec{F}$ along a displacement vector $\vec{D}$ is $W = \vec{F} \cdot \vec{D}$

**Example:** $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ moves a particle along a straight line from $P(2,1,0)$ to $Q(4,6,2)$. Find the work done.

$$\vec{D} = \overrightarrow{PQ} = \langle 4-2, 6-1, 2-0 \rangle = \langle 2, 5, 2 \rangle$$




$$\vec{F} = \langle 3, 4, 5 \rangle \qquad \vec{F} \cdot \vec{D} = \langle 3,4,5 \rangle \cdot \langle 2,5,2 \rangle$$

$$= 6 + 20 + 10 = 36 = \text{work.}$$

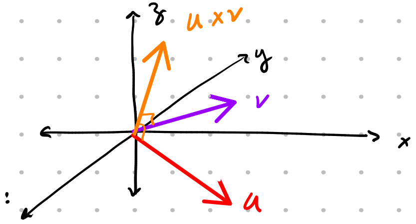
# Section 12.4: Cross Products

**Definition:** If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ & $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the cross product is given by:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2 b_3 - a_3 b_2) - \vec{j}(a_1 b_3 - a_3 b_1) + \vec{k}(a_1 b_2 - a_2 b_1)$$

$$= \langle a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1 \rangle$$

* This only applies to 3-dimensions.




**Theorem:** The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

**Proof sketch:** $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ in components.

Orthogonal $\Rightarrow$ $\theta = \pi/2$

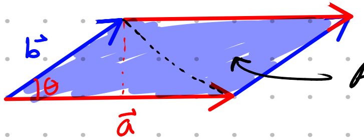
$$\boxed{\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}}$$

**Theorem:** If the angle $\theta$ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, then: $\boxed{|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta}$

**Corollary:** Two nonzero vectors are parallel if and only if $\vec{a} \times \vec{b} = 0$

- Relationship between cross products and area

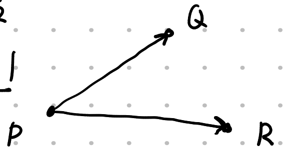
$$\sin\theta = \frac{y}{|\vec{b}|}$$




Area $= |\vec{a} \times \vec{b}|$

**Example:** What is the area of a triangle w/ vertices $P(1,4,6)$ $Q(-2,5,-1)$ $R(1,-1,1)$

1. Find $\vec{PQ} \times \vec{PR}$
2. Find $\dfrac{|\vec{PQ} \times \vec{PR}|}{2}$



## Properties of Cross Products

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ anti-commutativity

2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times c\vec{b}$

3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

left & right distributivity

* 5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ associativity.

6. $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ "Bac Cab"

Scalar triple product: $\vec{a} \cdot (\vec{b} \times \vec{c})$

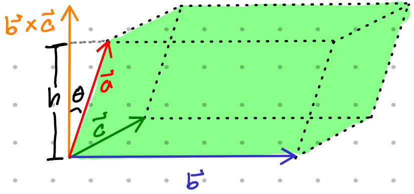
If you write this in components you will see

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

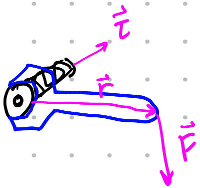
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{a}| \cdot |\vec{b} \times \vec{c}| \cdot \cos\theta$$

$\Rightarrow$ Scalar triple product $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ corresponds to the volume spanned by $\vec{a}, \vec{b}, \vec{c}$.

- Triple Products



□ Torque $\qquad \vec{\tau} = \vec{r} \times \vec{F}$ $\qquad$ Break: :57



# Section 12.3 Practice Problems

**1.** Which of the following expressions are meaningful? Which are meaningless? Explain.

(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ N

(b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$

(d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

(e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$

(f) $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$

a) Y N

b) Y N

c) Y N

d) Y N

e) Y N

f) Y N

53. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

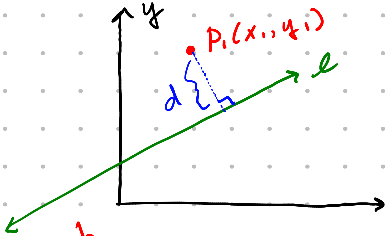
in $\mathbb{R}^2$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

$ax + by + c = 0$

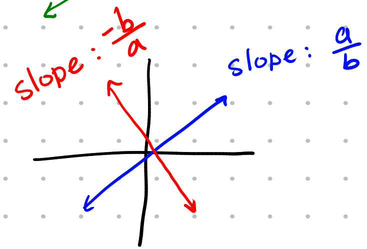
$\text{dist}(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

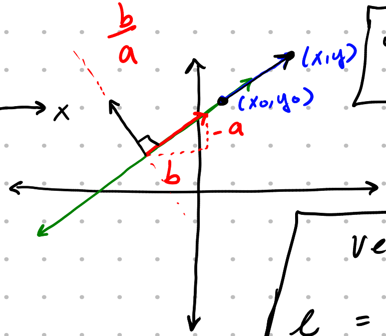
- Write in $y = mx + b$ form: $y = \frac{-ax - c}{b}$ $\quad x = \frac{-by - c}{a}$




Step 1: Write $ax + by + c = 0$ as a vector equation.

$$\boxed{y = \frac{-a}{b}x - c}$$

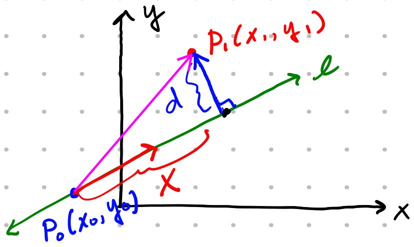



$\vec{s} = \langle b, -a \rangle$

Vector equation:

$$\ell = \{(x,y) \in \mathbb{R} : \langle a, b \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0\}$$




$$\ell = \{(x,y)\in\mathbb{R} : \langle a,b\rangle \cdot \langle x-x_0, y-y_0\rangle = 0\}$$

$$\overrightarrow{P_0P_1} = \langle x_1-x_0,\ y_1-y_0\rangle$$

$$\vec{v}_\ell = \langle b,-a\rangle$$

$$\vec{u}_\ell = \langle a,b\rangle$$

$$\boxed{\operatorname{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

$$\operatorname{Comp}_{\vec{u}_\ell}\overrightarrow{P_0P_1} = \frac{\langle x_1-x_0,\ y_1-y_0\rangle \cdot \langle a,b\rangle}{|\langle a,b\rangle|}$$

$$= \frac{ax_1-ax_0+by_1-by_0}{\sqrt{a^2+b^2}} = \frac{ax_1+by_1-(ax_0+by_0)}{\sqrt{a^2+b^2}}$$

$$ax_0+by_0+c=0$$

$$ax_0+by_0=-c$$

$$= \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}} \qquad \square$$

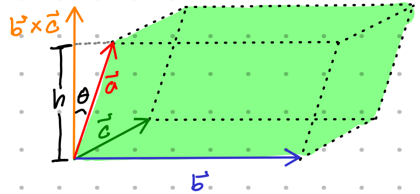
Answer: $\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

**54.** If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

# Section 12.4 Practice Problems

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$




**EXAMPLE 5** Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7\rangle$, $\mathbf{b} = \langle 2, -1, 4\rangle$, and $\mathbf{c} = \langle 0, -9, 18\rangle$ are coplanar.

$$\begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = \langle -18+36, -36, -18\rangle$$

$$= \langle 18, -36, -18\rangle$$

$$\vec{a}\cdot\langle 18, -36, -18\rangle = 18 - 4\cdot 36 + 7\cdot(-18)$$

$$18 - 8\cdot 18 + 7(18) = 0 \quad \checkmark$$

**9–12** Find the vector, not with determinants, but by using properties of cross products.

**9.** $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$ **10.** $\mathbf{k}\times(\mathbf{i}-2\mathbf{j})$

**11.** $(\mathbf{j}-\mathbf{k})\times(\mathbf{k}-\mathbf{i})$ **12.** $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})$

Properties of Cross Products

1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ anti-commutativity
2. $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times c\vec{b}$
3. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
4. $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$

$\Big\}$ left + right distributivity

* 5. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ associativity.

6. $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ "Bac Cab"

- $$(\vec{i}\times\vec{j})\times\vec{k} = -\vec{k}\times(\vec{i}\times\vec{j})$$
$$= \vec{i}(\underbrace{-\vec{k}\cdot\vec{j}}_{0}) - \vec{j}(\underbrace{-\vec{k}\cdot\vec{i}}_{0}) = 0$$

- $$(\vec{i}+\vec{j})\times(\vec{i}-\vec{j}) = \cancelto{0}{\vec{i}\times\vec{i}} + \vec{i}\times(-\vec{j}) + \vec{j}\times\vec{i} - \cancelto{0}{\vec{j}\times\vec{j}}$$
$$-\vec{i}\times\vec{j} + \vec{j}\times\vec{i} = \boxed{2\vec{j}\times\vec{i}}$$

**13.** State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ (b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (d) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

(e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ (f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

**44.** (a) Find all vectors $\mathbf{v}$ such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$

(b) Explain why there is no vector $\mathbf{v}$ such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

**46.** (a) Let $P$ be a point not on the plane that passes through the points $Q$, $R$, and $S$. Show that the distance $d$ from $P$ to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

**49.** Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

$$\underbrace{\cancel{\vec{a} \times \vec{a}}}_{0} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \underbrace{\cancel{\vec{b} \times \vec{b}}}_{0} = \vec{a} \times \vec{b} + \vec{a} \times \vec{b} = 2\,\vec{a} \times \vec{b} \quad \checkmark$$

# Participation Points Questions

1. What is the angle between $\vec{a} = \langle 1, 2 \rangle$ and $\vec{b} = \langle 3, 1 \rangle$?

2. Suppose $\vec{a} = \langle 1, 0, 1 \rangle$ and $\vec{b} = \langle 0, 2, 0 \rangle$. Compute $\vec{a} \times \vec{b}$