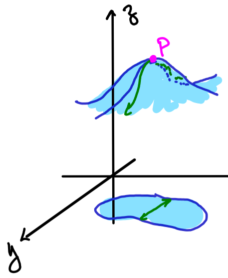
A "Proof" for the Second Derivative Test

(For graphs of functions $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^2$)

Ideas from Geometry:



- First, we would like to prove that the function f has a maximum (or minimum) at a point p iff the Hessian matrix of f at p is positive definite.

- Let $r(t) = r_0 + tv$, with $r_0 = (x_0, y_0)$ and $v = (a, b)$ be the vector equation of a line in $\mathbb{R}^2$ such that $f(r(0)) = p = (x_0, y_0)$

- Set $g(t) = f(r(t))$ = a single variable function

- Then $g' = \frac{d}{dt} f(x_0 + ta, y_0 + tb) = \langle f_x(x_0 + ta, y_0 + tb), f_y(x_0 + ta, y_0 + tb) \rangle \cdot \langle a, b \rangle = \nabla f \cdot v = 0$

  $\Rightarrow \langle f_x, f_y \rangle = 0$

- Next, note that $g(t)$ is "concave down" if $g''(t) < 0$.

$$g''(t) = \frac{d}{dt} \langle f_x(r(t)), f_y(r(t)) \rangle \cdot \langle a, b \rangle = \langle f_{xx}(r(t)) \cdot a + f_{xy}(r(t)) \cdot b,\ f_{yx}(r(t)) \cdot a + f_{yy}(r(t)) \cdot b \rangle \cdot \langle a, b \rangle$$

$$= (a, b) \left. \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \right|_p \begin{pmatrix} a \\ b \end{pmatrix} < 0$$

- $f(x,y)$ itself is only "concave down" if p has a maximum, or in otherwords, if the above is true $\forall$ vectors (a, b). $\Rightarrow$ if the Hessian is negative definite.

- On the other hand, $f(x,y)$ is "concave up" and has a minimum if the Hessian is positive definite.

- f has a saddle point at p if some vectors give > 0 + others < 0.

How does this connect to the "usual" 2nd derivative test involving the determinant?

Fact: A matrix is positive definite iff all its k^{th} leading $k \times k$ minors have a positive determinant. It is negative definite iff its k^{th} leading $k \times k$ minors are not zero and are not all positive. (This is apparently known as Sylvester's criterion).

For us, we only have two minors: f_{xx} and the matrix itself. This immediately yields the "usual" form of the 2nd determinant test.

The conditions on the Hessian above generalize to arbitrary dimensions.