FINAL EXAM

MTH 164 Summer Session B

Friday, August 7, 2020

Instructions

This exam has two parts. Each part contains ten questions. You must complete both parts of the exam! If you score better on part A than either of your midterms, your score on part A will replace your lowest midterm score.

- **ABSOLUTELY NO COLLABORATION IS ALLOWED ON THIS EXAM**. You are not to communicate in any way with your fellow students during this exam.
- You must justify all your work completely. No credit will be given to answers without justification.
- You may **not** access or use solutions from the website "Math Stack Exchange", Chegg, Bartleby, or any other "homework help" website. If your solutions are suspiciously similar to a solution I find on one of these websites, lacks full justification, *and* uses methods that were not covered in this class, I may not give you credit. I might even ask you to give me a verbal explanation of your solution after the exam in order to see if you can produce the solution in your own words, with a good understanding.
- You **may** look at your textbook and notes during the exam. You are even allowed to use calculators such as Wolfram Alpha and Desmos as long as you write full justifications for your solutions that demonstrate you understand them. You may use theorems from the textbook without proof but you must prove results stated in exercises or examples if you use them.
- You will write your solutions to the below problems on paper, in a digital writing program, or type them using Latex. If you are writing your solutions on paper, please scan them using a scanning phone app or regular scanner. If you are using a digital writing program, please export your work as a pdf file and attach it to an email. **Write one problem per page** and clearly label which problem goes with what work.
- You have **3 hours after the time you receive the exam pdf document** to complete this exam. You will have 20 minutes to then send me your solutions in an email. The total time from when you receive the exam to when I receive your completed work is 3 hours and 20 minutes. This means I should **have your solutions in my inbox by the end of this time period**.
- **Please send your solutions in pdf format as an attachment to an email**. Pictures will be accepted in an emergency if your scanning methods fail you. **Put your name in the filename**. Send your solutions in the order they appear in this exam.

PART A

Problem 1

One of the below integrals exists while the other doesn't. Find out which one exists and compute its limit. Prove that the other does not exist.

a) $\lim_{(x,y)\to(0,0)} \frac{x^2+200y^2}{x^2+y^2}$

b) $\lim_{(x,y)\to(0,0)} x\sin(1/y)$

a) when $x=y$, $\frac{201x^2}{2x^2} = \frac{201}{2} \Rightarrow \lim_{(x,x)\to(0,0)} \frac{x^2+200y^2}{x^2+y^2} = \frac{201}{2}$

when $y=0$, $\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1 \Rightarrow$ DNE

b) $|x\sin(1/y)| \le |x| \Rightarrow \lim_{(x,y)\to(0,0)} x\sin(1/y) = 0$

Problem 2

Show that the point $(0,0)$ is a critical point of the function $f(x,y) = 4x^2y - x^3y - x^2y^2$ and prove that it is a local minimum.

Hint: One way to do this is to show that for all (x,y) inside the unit circle, $f(x,y) \geq f(0,0)$.

- $f_x(x,y) = 8xy - 3x^2y - 2xy^2$

  $f_y(x,y) = 8x^2 - x^3 - x^2 = 7x^2 - x^3$

  Clearly the point $(0,0)$ satisfies these two equations.

- Also, $f(0,0) = 0$

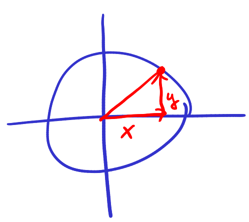
  claim: $4x^2y - x^3y - x^2y^2 \geq 0$ $\quad \forall (x,y) \in S = \{(x,y) : x^2+y^2 \leq 1\}$

  $$x^2y(4 - x - y) \geq 0$$

  $$\iff 4 \geq x+y$$

  For $x, y \in S$, the largest x, y can be is 1

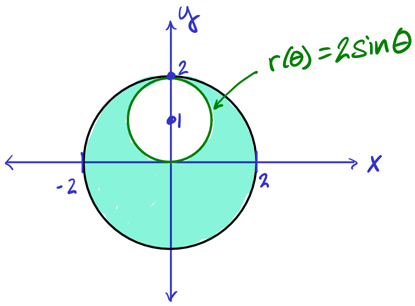
  $\Rightarrow$ $x+y \leq 2$ which is certainly < 1.



Problem 3

Set up but **do not evaluate** the integral representing the volume of the solid contained **inside** the sphere $4 = x^2 + y^2 + z^2$ and **outside** the cylinder $x^2 + (y-1)^2 = 1$

Hint: It might be helpful to think of this volume as a *double integral* although this is not strictly necessary.



$$z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$$

$$\int_0^{2\pi}\int_0^2 \sqrt{4-r^2}\, r\,dr\,d\theta - \int_0^{\pi}\int_0^{2\sin\theta} \sqrt{4-r^2}\, r\,dr\,d\theta$$

Problem 4

Find the **parametric equations** representing the curve (or curves) of intersection of the surface

$$\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$$

and the plane $x = z$. Describe the curve (or curves) in words (is it a circle, parabola, line, ...).

when $x = z$, we have

$\frac{x^2}{4} - \frac{x^2}{4} + y^2 = y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow$ Since there are no more constraints, our parametric equations are

$\alpha_1(t) = \langle t, 1, t \rangle$ and $\alpha_2(t) = \langle t, -1, t \rangle$

α_1 & α_2 are <u>lines</u>

Problem 5

Find the **extreme values** of $f(x,y) = e^{-xy}$ on the region described by $x^2 + 4y^2 \leq 1$.

- $f_x(x,y) = -x e^{-xy}$
  $f_y(x,y) = -y e^{-xy}$ $\Rightarrow$ $(x,y) = (0,0)$ is the only solution to this.

- $f(0,0) = 1$. Note that $e^{-xy} = \frac{1}{e^{xy}}$. when $x = y = \frac{1}{4}$, $\frac{1}{16} + \frac{1}{4} \leq 1$

  But $\frac{1}{e^{1/16}} \approx .9 < 1$

  On the other hand, if $x = -\frac{1}{4}$ & $y = \frac{1}{4}$, $e^{-xy} = e^{1/16} \approx 1.06 > 1$

  $\Rightarrow$ $(0,0)$ is niether a max or a min.

- Let $x = \cos\theta$, $y = \frac{1}{2}\sin\theta$. then $f(x,y) = f(\theta) = e^{-\frac{1}{2}\cos\theta\sin\theta} = e^{-\frac{1}{4}\sin(2\theta)}$

$$\frac{d}{d\theta} e^{-\frac{1}{4}\sin(2\theta)} = e^{-\frac{1}{4}\sin(2\theta)} \cdot \frac{d}{d\theta}\left(-\frac{1}{4}\sin(2\theta)\right) = -\frac{1}{2} e^{-\frac{1}{4}\sin(2\theta)}(\cos(2\theta))$$

& this $= 0$ $\Leftrightarrow$ $\cos(2\theta) = 0$ $\Leftrightarrow$ $2\theta = \pi/2$ or $3\pi/2$

$$2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4} \qquad 2\theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{4}$$

thus the critical points are

$$e^{-\frac{1}{4}\sin(\pi/2)} \quad \& \quad e^{-\frac{1}{4}\sin(3\pi/2)}$$

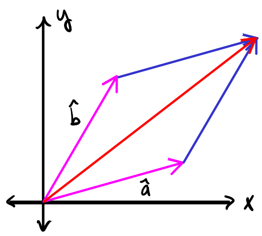
$$\underset{\text{min}}{e^{-1/4} \approx .78} \quad < \quad \underset{\text{max}}{e^{1/4} \approx 1.3}$$

Problem 6

If $2\vec{w} = |\vec{u}|\vec{v} + |\vec{v}|\vec{u}$, where $\vec{w}, \vec{u}$, and $\vec{v}$ are all nonzero vectors in $\mathbb{R}^3$, show that $\vec{w}$ bisects the angle between $\vec{u}$ and $\vec{v}$. (Also $\vec{u} \neq \vec{v}$). In other words show that the angle between $\vec{u}$ and $\vec{w}$ is equal to the angle between $\vec{v}$ and $\vec{w}$.

Hint: Start by factoring out a $|\vec{a}||\vec{b}|$.

- Note that $2\vec{c} = |\vec{a}||\vec{b}|\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$



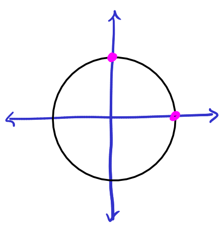
$\Rightarrow \vec{c}$ is a scalar multiple of a vector bisecting $\angle ab$.

Problem 7

What is the length of the shortest curve between the two points $(1,0,0)$ and $(0,1,1)$ on the cylinder $x^2 + y^2 = 1$? What are the parametric equations of this curve? Curves like this which minimize distance are called *geodesics*.

Hint: You may assume that this curve is part of a *helix*. To see why, imagine cutting the cylinder along a single vertical line and unwrapping it so that it forms a rectangle. Then the shortest path between the two points is a straight line. When you glue the rectangle back together, the straight line will be part of a helix.

- A helix on the surface of the cylinder has the form $\langle \cos t, \sin t, bt \rangle$ for some constant b.
- $\langle \cos t_0, \sin t_0, bt_0 \rangle = \langle 1, 0, 0 \rangle \iff t_0 = 0$
- $\langle \cos t_f, \sin t_f, bt_f \rangle = \langle 0, 1, 1 \rangle \iff bt_f = 1, \ \sin(t_f) = 0$



the projection of the points onto the xy-plane shows that $t_f = \pi/2 \implies b = \frac{2}{\pi}$

$$|r'(t)| = \sqrt{1 + \frac{4}{\pi^2}}$$

- So $L = \int_0^{\pi/2} \sqrt{1 + \frac{4}{\pi^2}}\, dt = \sqrt{1 + \frac{4}{\pi^2}}\left(\pi/2\right)$

Problem 8

The function $f(x, y) = ln(x^2 + y^2)$ defined on $\mathbb{R}^2 - \{0\}$ describes the electric potential due to a line charge. Show that it satisfies Laplace's equation. (Laplace's equation is $\nabla^2 f = 0$ and can be found in a different form on page 920).

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{-2x^2+2y^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = 0$$

PART B

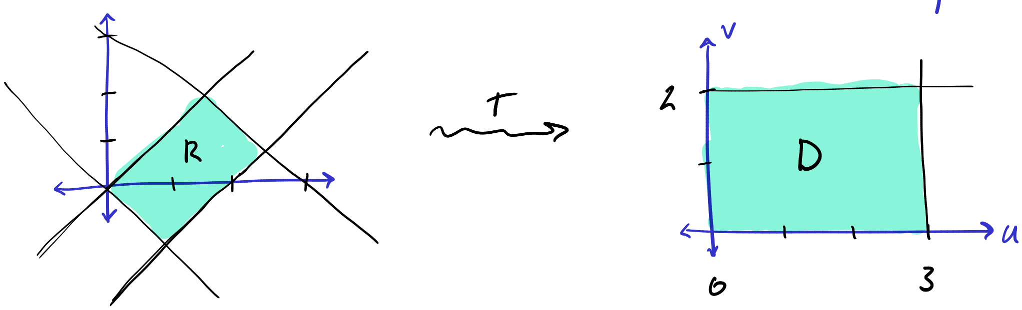
Problem 1

Evaluate the integral. You may find it helpful to make a change of coordinates.

$$\int\int_R (x+y)e^{x^2-y^2}\,dA$$

where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$

let $u = x+y$, $v = x-y$ then $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$ $\Rightarrow$ $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{2}$



$\Rightarrow$ we have $-\frac{1}{2}\int_0^3\int_0^2 u e^{uv}\,dv\,du = \frac{-1}{2}\int_0^3 e^{uv}\Big|_0^2 du = -\frac{1}{2}\int_0^3 (e^{2u}-1)\,du$

$= \frac{-1}{2}\left(\frac{1}{2}e^{2u} - u\right)\Big|_0^3 = \frac{-1}{2}\left(\frac{1}{2}e^6 - 3\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{-1}{4}e^6 + \frac{3}{2} + \frac{1}{4}$

$= \frac{-e^6+7}{4}$

Problem 2

The vector field $\vec{F}(x,y) = \left\langle \frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}} \right\rangle$ describes the gravitational force field of an object with mass 1 and gravitational constant 1. Find the gravitational potential function. (Find a function f such that $\nabla f = \vec{F}$)

$$\int \frac{-x}{(x^2+y^2)^{3/2}}\,dx =$$

Let $u = x^2+y^2$ $\quad du = 2x\,dx \quad -\frac{1}{2}du = -x\,dx$

$$-\frac{1}{2}\int u^{-3/2}du = (-\tfrac{1}{2})(-2)\,u^{-1/2} = u^{-1/2} = \frac{1}{\sqrt{x^2+y^2}} + C(y)$$

$$\Rightarrow \quad f = \frac{1}{\sqrt{x^2+y^2}}$$

Similarly, $\displaystyle\int \frac{-y}{(x^2+y^2)^{3/2}}\,dy = \frac{1}{\sqrt{x^2+y^2}} + D(x)$

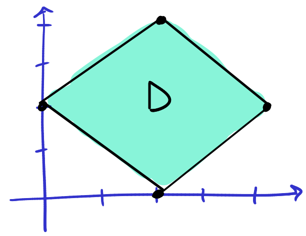
Problem 3

Compute the integral $\int_C \vec{F} \cdot dr$ where C is the positively oriented square with vertices $(2,0)$, $(0,2)$, $(2,4)$, and $(4,2)$ and $\vec{F} = \langle -\cos x \cos y + \frac{1}{2}xy^2, \sin x \sin y + \frac{1}{2}x^2 y + 5x \rangle$.

P Q

Hint: You probably want to use the theorem that converts line integrals into double integrals...

Green's Theorem!



$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$\frac{\partial P}{\partial y} = \cos x \sin y + xy$ $\qquad$ $\frac{\partial Q}{\partial x} = \cos x \sin y + xy + 5$

$2^2 + 2^2 = 8$

$$\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 5 \quad \Rightarrow \int_C \vec{F} \cdot d\vec{r} = 5 \iint_D dA = 5 \, \text{Area}(D) = 5(\sqrt{8}^2) = 40$$

Problem 4

Is there a vector field $\vec{G}$ on $\mathbb{R}^3$ such that $\text{curl}(\vec{G}) = \langle x^3, y, z \rangle$? If yes, find an example. If no, explain why not.

No theorem: $\nabla \cdot (\nabla \times G) = 0$

But if $\nabla \times G = \langle x^3, y, z \rangle$, then

$\nabla \cdot \langle x^3, y, z \rangle = 3x^2 + 1 + 1 \geq 0 \Rightarrow$ Contradiction.

Problem 5

Evaluate $\int\int_S (\nabla \times \vec{F}) \cdot dS$ where $\vec{F} = \langle 2yz, -x+3y, x^2+z \rangle$ and S is the cylinder (without top or bottom) $x^2+y^2=1$ with $0 \le z \le 1$.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 2yz & -x+3y & x^2+z \end{vmatrix} = i(0-0) - j(2x-2y) + k(-1-2z)$$

$$= \langle 0, -2x+2y, -1-2z \rangle$$

Parameterize:

$x = \cos t$

$y = \sin t$

$z = z$

$$\begin{vmatrix} & i & j & k \\ r_t = & \langle -\sin t, & \cos t, & 0 \rangle \\ r_z = & \langle 0, & 0, & 1 \rangle \end{vmatrix} = \langle \cos t, \sin t, 0 \rangle = \langle r_t \times r_z \rangle$$

$$\Rightarrow (\nabla \times F) \cdot \langle r_t \times r_z \rangle = (\sin t)(-2\cos t) + (\sin t)(2\sin t) = -2\cos t \sin t + 2\sin^2 t$$

$$\int_0^{2\pi}\int_0^1 (-2\cos t \sin t)\, dz\, dt + \int_0^{2\pi}\int_0^1 2\sin^2 t\, dz\, dt = 2\pi$$

Problem 6

Find the positively oriented simple closed curve C that maximizes the integral $\int_C (y^3 - y)dx - 2x^3 dy$.

Green's Theorem says:

$$\int_C \overset{P}{(y^3-y)}dx - \overset{Q}{2x^3}dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$\frac{\partial P}{\partial y} = 3y^2 - 1$ $\qquad$ $\frac{\partial Q}{\partial x} = -6x^2$ $\quad \Rightarrow \quad \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -6x^2 - 3y^2 + 1 = f(x,y)$

$f(x,y) = 0 \quad \Rightarrow -6x^2 - 3y^2 + 1 = 0 \iff 6x^2 + 3y^2 = 1$

This is the equation of an ellipse

Furthermore $f(0,0) = 1 > 0 \quad \Rightarrow f(x,y) > 0 \ \forall \ x,y \in$ ellipse.

$C(t) = \left\langle \frac{1}{\sqrt{6}} \cos t, \frac{1}{\sqrt{3}} \sin t \right\rangle$

Problem 7

Find the surface area of the helicoid, which has parametric equations $\vec{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$ where $0 \le u \le 1$ and $0 \le v \le 3\pi$.

$$\begin{vmatrix} & i & j & k \\ \vec{r}_u = & \langle \cos v, & \sin v, & 0 \rangle \\ \vec{r}_v = & \langle -u\sin v, & u\cos v, & 1 \rangle \end{vmatrix} = \langle \sin v, -\cos v, u\cos^2 v + u\sin^2 v \rangle$$

$$= \langle \sin v, -\cos v, u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1+u^2}$$

$$A(S) = \int_0^1 \int_0^{3\pi} \sqrt{1+u^2}\, dv\, du = (3\pi) \int_0^1 \sqrt{1+u^2}\, du$$

$$= 3\pi \int_{u=0}^{u=1} \sqrt{1+\tan^2 x}\, \sec^2 x\, dx = 3\pi \int_{u=0}^{u=1} \sec^3 x\, dx = 3\pi \int_{u=0}^{u=1} \sec x \cdot \sec^2 x\, dx$$

Integration by Parts

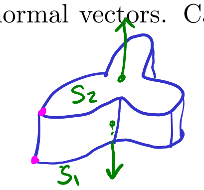
$$= 3\pi \left(\sec\theta \tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta \right)$$

$$= 3\pi \left(\frac{1}{2} u \sqrt{1+u^2} + \frac{1}{2} \ln \left| u + \sqrt{1+u^2} \right| \right) = 3\pi \left(\frac{1}{2}\sqrt{2} + \frac{1}{2} \ln(1+\sqrt{2}) \right)$$

Problem 8

Let S be the "cylinder" whose base, in the xy-plane, is bounded by the polar curve $r(\theta) = 3 - 2\cos(3\theta)$ and whose top is bounded by the same curve in the plane $z = 1$. Let S_1 be the base, S_2 the top, and S_3 the curved side parallel to the z-axis. Orient S so that it has outward pointing normal vectors. Calculate $\int\int_{S_3} \vec{F} \cdot d\vec{S}$ where $\vec{F}$ is:

$$\vec{F}(x, y, z) = \left\langle -y^2x + z^{10}, \frac{1}{3}y^3 - 2zy, z^2 \right\rangle$$



We will use divergence theorem

$$\nabla \cdot \vec{F} = -y^2 + y^2 - 2z + 2z = 0 \quad \Rightarrow \quad \iint_{S_3} = -\underbrace{\iint_{S_1}}_{A} - \underbrace{\iint_{S_2}}_{B}$$

$$A: -\iint_{S_2} F \cdot dr = -\iint_D F \cdot (\vec{r}_x \times \vec{r}_y)\, dx\, dy = -\int_0^{2\pi} \int_0^{3-2\cos(3\theta)} F \cdot \langle 0,0,1 \rangle\, r\, dr\, d\theta$$

$= -\int_0^{2\pi} \int_0^{3-2\cos(3\theta)} z^2 \cdot r\, dr\, d\theta$ But $z = 1$ in S_2 $\Rightarrow$ we have

$$-\int_0^{2\pi} \int_0^{3-2\cos(3\theta)} r\, dr\, d\theta = -\frac{1}{2} \int_0^{2\pi} (3 - 2\cos(3\theta))^2\, d\theta = \left(-\frac{1}{2}\right)(22\pi) = -11\pi$$

For B, we have $-\int_0^{2\pi} \int_0^{3-2\cos(3\theta)} F \cdot \langle 0,0,-1 \rangle\, r\, dr\, d\theta$

$= -\int_0^{2\pi} \int_0^{3-2\cos(3\theta)} z^2 \cdot r\, dr\, d\theta$ But $z = 0$ in S_1! $\Rightarrow$ we just have zero.

$$\Rightarrow \iint_{S_3} \vec{F} \cdot d\vec{S} = -11\pi$$