Wednesday, June 30, 2021

MTH 164 Lecture Notes

Review Day 3: Last Day

The divergence theorem: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\, \vec{F}\, dV$

Problem 1: Find the flux of the vector field $F(x,y,z) = \langle z, y, x\rangle$ over the unit sphere $x^2+y^2+z^2=1$

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x} z + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} x = 1$$

$$\iiint_E dV = \frac{4\pi}{3} \leftarrow \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

Problem 2: Consider the vector field which describes the electric field of a charge Q located at the origin. ($\vec{x} = \langle x,y,z\rangle$ is a position vector)

$$\vec{E} = \frac{\varepsilon Q}{|\vec{x}|^3}\cdot \vec{x}$$

Use the divergence theorem to show that the electric flux of E through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \vec{E}\cdot d\vec{S} = 4\pi\varepsilon Q \qquad \text{Gauss' Law}$$

$$\vec{E} = \frac{\varepsilon Q}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$

$$\text{div}\,\vec{E} = \vec{\nabla}\cdot\vec{E} = \varepsilon Q\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)\right]$$

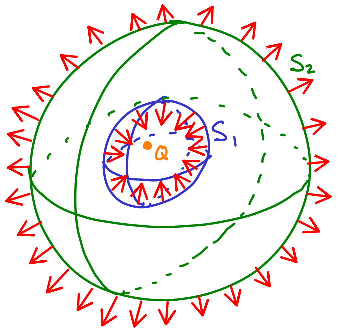
$$= \varepsilon Q \; \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)\cdot x}{(x^2+y^2+z^2)^{3}}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}(2y)\cdot y}{(x^2+y^2+z^2)^{3}}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}(2z)\cdot z}{(x^2+y^2+z^2)^{3}}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}\left[3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2\right]}{(x^2+y^2+z^2)^3} = \frac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{1/2}} = 0$$

$$\Rightarrow \iiint_E \operatorname{div}\vec F\, dV = \iint_S \vec F\cdot d\vec S = \iint_S \vec F\cdot\vec n\, dS = \iint_{S_1} \vec F\cdot(-n_1)\, dS + \iint_{S_2} F\cdot(\vec n_2)\, dS$$



$$= -\iint_{S_1} \vec F\cdot d\vec S + \iint_{S_2} \vec F\cdot d\vec S$$

$\iint_{S_2} \vec E\cdot d\vec S = 4\pi\varepsilon Q$ We know $\operatorname{div}\vec E = 0$

$$\iiint_E \operatorname{div}\vec E\, dV = 0 = -\iint_{S_1} \vec E\cdot d\vec S + \iint_{S_2} \vec E\cdot d\vec S$$

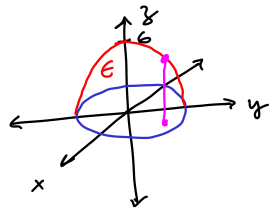
$$\iint_{S_1} \vec E\cdot d\vec S = \iint_{S_2} \vec E\cdot d\vec S$$

S_1 is a sphere of radius a.

$$\iint_{S_1} \vec E\cdot d\vec S = \iint_{S_1} \vec E\cdot\vec n\, dS = \frac{\varepsilon Q}{a^2}\iint_{S_1} d\vec S = \frac{\varepsilon Q}{a^2}\cdot 4\pi a^2 = 4\pi\varepsilon Q \quad \text{as desired.}$$

Problem 3: Use the divergence theorem to evaluate $\iint_S \vec F\cdot d\vec S$ where $\vec F = \langle 2xz, 1-4y^2, 2z-z^2\rangle$ and S is the solid bounded by $z = 6-2x^2-2y^2$ & the plane $z=0$ $\boxed{18\pi}$

$$\iint_S \vec F\cdot d\vec S = \iiint_E \operatorname{div}\vec F\, dV$$



$$\vec\nabla\cdot\vec F = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(1-4y^2) + \frac{\partial}{\partial z}(2z-z^2)$$

$$= 2z - 8y + 2 - 2z = \boxed{-8y+2}$$

when $z=0$, S is a circle $6 = 2x^2+2y^2$ of radius $\sqrt{3}$

$3 = x^2+y^2$

$$\iiint_E (-8y+2)\, dV = \int_0^{2\pi}\int_0^{\sqrt3}\int_0^{6-2r^2} r(-8r\sin\theta+2)\, dz\, dr\, d\theta$$

$-8r^2\sin\theta + 2r$

$$\int_0^{2\pi}\int_0^{\sqrt3} (6-2r^2)(-8r^2\sin\theta+2r)\, dr\, d\theta$$

$$= \boxed{\int_0^{2\pi}\int_0^{\sqrt3} (-48r^2\sin\theta - 4r^3 + 12r + 16r^4\sin\theta)\, dr\, d\theta}$$

Problem 4: Evaluate $\int_C \ln y\, e^{-x} dx - \frac{e^{-x}}{y} dy + z\, dz$ where C is given by
$\vec{r}(t) = (t-1)\,i + e t^4\, j + (t^2+1)\, k$ for $0 \le t \le 1$

$$\int_C \nabla f \cdot ds = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\vec{F} = \left\langle \ln y\, e^{-x}, -\frac{e^{-x}}{y}, z \right\rangle$$

$f_x(x,y,z) = \ln y\, e^{-x} \rightsquigarrow f(x,y,z) = -\ln(y)\cdot e^{-x} + K_1(y,z)$

$f_y(x,y,z) = -\frac{e^{-x}}{y} \rightsquigarrow f(x,y,z) = -\ln(y)\cdot e^{-x} + K_2(x,z)$

$f_z(x,y,z) = z \rightsquigarrow f(x,y,z) = \frac{1}{2}z^2 + K_3(x,y)$

$\therefore f(x,y,z) = -\ln(y)\cdot e^{-x} + \frac{1}{2}z^2 + K$ + $\vec{F}$ is conservative

$\vec{r}(0) = \langle -1, 1, 1\rangle$
$\vec{r}(1) = \langle 0, e, 2\rangle$
$\Rightarrow \int_C \vec{F} = f(-1,1,1) - f(0,e,2) = \frac{1}{2} - (-1+2) = \frac{1}{2} - 1$

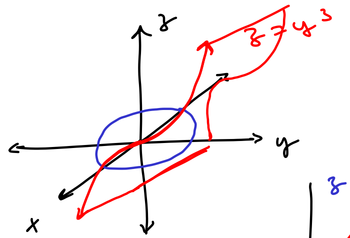
$\iint_D f(x,y)\, dA$

Question 8 (16.4):

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$\int_C \underset{P}{y^4\, dx} + \underset{Q}{2xy^3\, dy}$ C is the ellipse $x^2 + 2y^2 = 2$

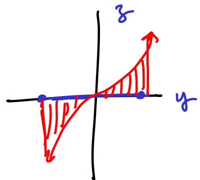
$f(x,y) = y^3$



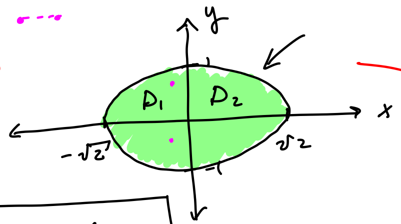
$\frac{\partial Q}{\partial x} = 2y^3$

$\frac{\partial P}{\partial y} = 4y^3$

$\Rightarrow \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 2y^3 - 4y^3 = -2y^3$

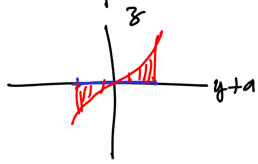


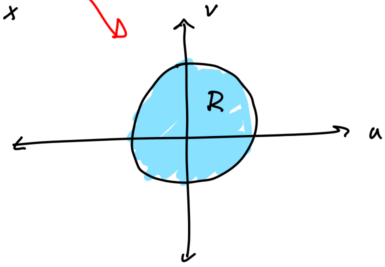
$\iint_D (-2y^3)\, dA$ $= 0$
↳ $f(x,y)$



T^{-1}

$\left(\frac{x}{\sqrt{2}}\right)^2 + y^2 = 1$





Let $x = u$
$y = \frac{1}{\sqrt{2}}\cdot v$

$(x,y) \rightsquigarrow (-x,y)$
$f(x,y) = -f(x,-y)$

$x^2 + 2y^2 = 2 \rightsquigarrow u^2 + v^2 = 2$

$u = r\cos\theta$
$v = r\sin\theta$

$$|J| = \begin{vmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \end{vmatrix} = \frac{1}{\sqrt{2}}$$

$$\iint_R \left(-2\cdot\frac{1}{2^{3/2}}\cdot v^3\right)\cdot\frac{1}{2^{1/2}}\, dA = \left(\frac{-2}{2^{3/4}}\right)\int_0^{2\pi}\int_0^{\sqrt{2}} r^4 \sin^3\theta\, dr\, d\theta$$

Suggested Problems.

Ch 16)

Use Div Thm to calculate $\iint_S \vec{F} \cdot d\vec{S}$

16.9)

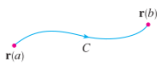
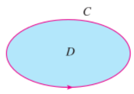
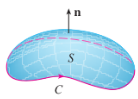
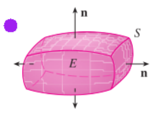
5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

6. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$,
S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are positive numbers

25–30 Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

25. $\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0$, where $\mathbf{a}$ is a constant vector

26. $V(E) = \frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_E \text{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

← will be on exam.

16.8)

2–6 Use Stokes' Theorem to evaluate $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = x^2 \sin z\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$,
S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward

3. $\mathbf{F}(x, y, z) = ze^y\,\mathbf{i} + x\cos y\,\mathbf{j} + xz\sin y\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geqslant 0$, oriented in the direction of the positive y-axis

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\,\mathbf{i} + x^2y\,\mathbf{j} + x^2z^2\,\mathbf{k}$,
S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + xy\,\mathbf{j} + x^2yz\,\mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1, \pm1, \pm1)$, oriented outward

6. $\mathbf{F}(x, y, z) = e^{xy}\,\mathbf{i} + e^{xz}\,\mathbf{j} + x^2z\,\mathbf{k}$,
S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis

16.7) Evaluate the surface integral

9. $\iint_S x^2yz\,dS$,
S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

10. $\iint_S xz\,dS$,
S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant

11. $\iint_S x\,dS$,
S is the triangular region with vertices $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 4)$

12. $\iint_S y\,dS$,
S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

13. $\iint_S z^2\,dS$,
S is the part of the paraboloid $x = y^2 + z^2$ given by $0 \leqslant x \leqslant 1$

14. $\iint_S y^2z^2\,dS$,
S is the part of the cone $y = \sqrt{x^2 + z^2}$ given by $0 \leqslant y \leqslant 5$

15. $\iint_S x\,dS$,
S is the surface $y = x^2 + 4z$, $0 \leqslant x \leqslant 1$, $0 \leqslant z \leqslant 1$

16. $\iint_S y^2\,dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

17. $\iint_S (x^2z + y^2z)\,dS$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

22. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$,
S is the helicoid of Exercise 7 with upward orientation

23. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

24. $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation

25. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k}$, S is the sphere with radius 1 and center the origin

26. $\mathbf{F}(x, y, z) = y\,\mathbf{i} - x\,\mathbf{j} + 2z\,\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$, oriented downward

27. $\mathbf{F}(x, y, z) = y\,\mathbf{j} - z\,\mathbf{k}$,
S consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

28. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$,
S is the surface $z = x \sin y$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant \pi$, with upward orientation

29. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the cube with vertices $(\pm1, \pm1, \pm1)$

30. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$, S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

31. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\mathbf{j} + z^2\,\mathbf{k}$, S is the boundary of the solid half-cylinder $0 \leqslant z \leqslant \sqrt{1 - y^2}$, $0 \leqslant x \leqslant 2$

32. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$,
S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

Section 16.5

23–29 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\,\mathbf{F}(x, y, z)$$
$$(\mathbf{F} \cdot \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z)$$
$$(\mathbf{F} \times \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)$$

23. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

24. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$

25. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F} \cdot \nabla f$

26. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f) \times \mathbf{F}$

27. $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl}\mathbf{F} - \mathbf{F} \cdot \operatorname{curl}\mathbf{G}$

28. $\operatorname{div}(\nabla f \times \nabla g) = 0$

29. $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$

- 5 Part A (Ch. 16)
- 6 Part B (before Ch. 16)

$\int_C y^4 \cdot dx + 2xy^3 dy$ $\quad r(t) = \langle \sqrt{2}\cos t, \sin t \rangle$ $\quad x^2 + 2y^2 = 2$

$x'(t) = -\sqrt{2} \cdot \sin t$

$y'(t) = \cos t$

$\int f(t)\, ds = \int_a^b f(\vec{r}(t)) \cdot |\vec{r}'(t)|\, dt$

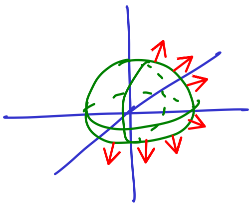
$-\sqrt{2}\int_0^{2\pi} \sin^5 t \cdot dt + 2\sqrt{2}\int_0^{2\pi} \cos^2 t \sin^3 t\, dt$

Problem 13 WW5

$M = x^2 + y^2 + 4z^2 = 1$ $\quad z \le 0$

$\iint_M (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$

$\vec{F}(x,y,z) = \langle y, -x, zx^3y^2 \rangle$



$4z^2 = 1 - x^2 - y^2$

$z = -\frac{1}{2}\sqrt{1 - x^2 - y^2}$

$f(x,y,z) = -\frac{1}{2}\sqrt{1 - x^2 - y^2} - z$

$V(x,y) = (\vec{\nabla} \times \vec{F}) \cdot (\vec{\nabla} f)$

$\vec{r}_u \times \vec{r}_v$

$\vec{\nabla} f \cdot dA = d\vec{S}$

$(\vec{r}_u \times \vec{r}_v) \cdot dA = d\vec{S}$