Tuesday, June 29, 2021

MTH 164 Lecture Notes

Review day 2

$ds = |\vec{r}'(t)|dt$ $\qquad$ $\vec{r}(t) = \langle x_1(t), \ldots, x_n(t) \rangle$

line segment between P_0 + P_1: $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$

Line Integrals: $\int_a^b f(\vec{r}(t))\,|\vec{r}'(t)|\,dt$ $\quad$ or $\quad$ $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

Problem 1: Evaluate $\int_C y\,ds$ where $C: x = t^2,\ y = 2t \quad 0 \le t \le 3$

$r(t) = \langle t^2, 2t \rangle$ $\qquad$ $u = 4t^2 + 4$

$r'(t) = \langle 2t, 2 \rangle$ $\qquad$ $du = 8t\,dt \quad \frac{1}{4}du = 2t\,dt$

$|r'(t)| = \sqrt{4t^2 + 4}$

$$\int_0^3 (2t) \cdot \sqrt{4t^2+4}\,dt = \frac{1}{4}\int_4^{40} u^{1/2}\,du$$

Problem 2: Evaluate $\int_C e^x\,dx$ where C is the arc of $x = y^3$ from $(-1,-1)$ to $(1,1)$

$\vec{r}(t) = \langle t^3, t \rangle \qquad -1 \le t \le 1$

$$\int_{-1}^1 e^{t^3} \cdot 3t^2\,dt \qquad 3t^2 = \frac{d}{dt}x(t)$$

Problem 3: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = \langle \sin x, \cos y, xz \rangle$ + C is the trace of

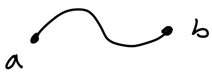
$\vec{r}(t) = t^3\vec{i} - t^2\vec{j} + t\vec{k} \qquad 0 \le t \le 1$

$\vec{F}(\vec{r}(t)) = \langle \sin(t^3), \cos(-t^2), t^4 \rangle$

$\vec{r}'(t) = \langle 3t^2, -2t, 1 \rangle$

$$\int_0^1 \left(3t^2 \cdot \sin(t^3) - 2t\cos(-t^2) + t^4\right)dt$$

$$\boxed{\text{FTLI: } \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(a)) - f(\vec{r}(b))}$$



Problem 4: a) Find f such that $\nabla f = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle = \vec{F}(x,y,z)$

b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $C: r(t) = \langle t^2+1, t^2-1, t^2-2t \rangle \quad 0 \le t \le 2$

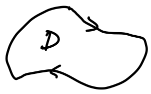
a)
$$\left.\begin{aligned} f_x(x,y,z) &= yze^{xz} &\rightsquigarrow\quad f(x,y,z) &= ye^{xz} + K_1(y,z) \\ f_y(x,y,z) &= e^{xz} &\longrightarrow\quad f(x,y,z) &= ye^{xz} + K_2(x,z) \\ f_z(x,y,z) &= xye^{xz} &\rightsquigarrow\quad f(x,y,\ ) &= ye^{xz} + K_3(x,y) \end{aligned}\right\} \quad \underline{f(x,y,z) = ye^{xz} + K}$$

b) $\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \nabla f \cdot \vec{r}\,'(t)\,dt = f(\vec{r}(0)) - f(\vec{r}(2)) = (-1 + K) - (3 + K) = -4$

$\vec{r}(0) = (1, -1, 0)$

$\vec{r}(2) = (5, 3, 0)$

$$\boxed{\text{Green's Theorem: } \int_C P\,dx + Q\,dy = \iint_D (\underbrace{Q_x - P_y}_{\vec{\nabla} \times F})\,dA}$$



Problem 5: Evaluate $\oint_C \underbrace{(3y - e^{\sin x})}_{P}dx + \underbrace{(7x + \sqrt{y^4 - 1})}_{Q}dy$ where C is the circle $x^2+y^2=9$

$\frac{\partial Q}{\partial x} = 7$

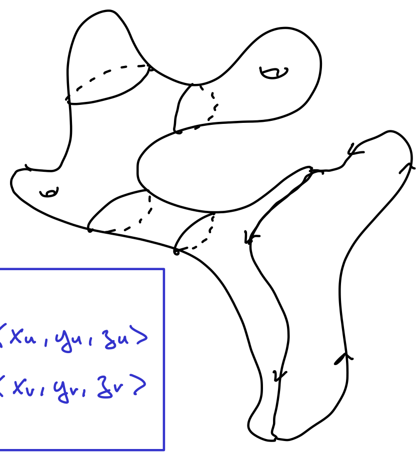
$\frac{\partial P}{\partial y} = 3$

$$\int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = \int_0^{2\pi} 4 \cdot \frac{r^2}{2}\Big|_0^3 \,d\theta = \int_0^{2\pi} 18\,d\theta = 18 \cdot \theta\Big|_0^{2\pi} = 36\pi$$

$D = \{(x,y) : x^2+y^2 \le 9\}$

πr^2

$$\boxed{\begin{array}{l} \text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} \quad \text{curl}(\nabla f) = 0 \quad \text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} \quad \text{div}(\text{curl } \vec{F}) = 0 \\[1em] \text{Green's theorem: } \oint_C \vec{F} \cdot d\vec{r} = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \vec{k}\,dA \end{array}}$$



1
$$\boxed{\begin{array}{ll} \text{For } S \text{ given by } \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle & \vec{r}_u = \langle x_u, y_u, z_u \rangle \\ \iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA & \vec{r}_v = \langle x_v, y_v, z_v \rangle \end{array}}$$

2
$$\boxed{\begin{array}{l} \text{For } S \text{ given by } z = g(x,y), \quad \vec{r}(x,y) = \langle x, y, g(x,y) \rangle \quad \begin{array}{l} r_x = \langle 1, 0, z_x \rangle \\ r_y = \langle 0, 1, z_y \rangle \end{array} \\ \iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y)) \sqrt{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1}\,dA \end{array}}$$

Problem 6: Evaluate $\iint_S x^2\,dS$ where S is the unit sphere $x^2+y^2+z^2=1$

$$r(\varphi,\theta) = \langle \sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi \rangle$$

$$\begin{vmatrix} i & j & k \\ r_\varphi = \langle \cos\varphi\cos\theta, & \cos\varphi\sin\theta, & -\sin\varphi \rangle \\ r_\theta = \langle -\sin\varphi\sin\theta, & \sin\varphi\cos\theta, & 0 \rangle \end{vmatrix} = i(\sin^2\varphi\cos\theta) + j(\sin^2\varphi\sin\theta) + k(\underline{\cos\varphi\sin\varphi}\cos^2\theta + \cos\varphi\sin\varphi\sin^2\theta)$$

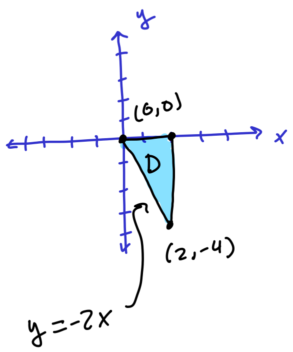
$$= \langle \sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \cos\varphi\sin\varphi \rangle$$

$$|r_\varphi \times r_\theta| = \sqrt{(\sin^2\varphi\cos\theta)^2 + (\sin^2\varphi\sin\theta)^2 + (\cos\varphi\sin\varphi)^2}$$

$$= \sqrt{\sin^4\varphi + \cos^2\varphi\sin^2\varphi} = \sqrt{\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)} = \sin\varphi$$

$$\int_0^{2\pi}\int_0^{\pi} (\sin^2\varphi\cdot\cos^2\theta)\cdot\sin\varphi\ d\varphi\, d\theta$$

Problem 7: Evaluate $\iint_S (z+3y-x^2)\,dS$ where S is the portion of $z = 2-3y+x^2 = g(x,y)$ that lies over the triangle in the xy-plane with vertices $(0,0)$, $(2,0)$, $(2,-4)$.



For S given by $z = g(x,y)$,

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1}\ dA$$

$$f(x,y,g(x,y)) = 2-3y+x^2+3y-x^2 = 2$$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = -3$$

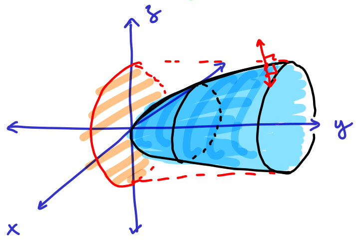
$$dS = \sqrt{4x^2+9+1}\ dA$$

$$\int_0^2\int_{-2x}^{0} 2\sqrt{4x^2+9+1}\ dy\,dx$$

$$\boxed{\text{Flux: } \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA}$$

$$dS = \underbrace{|r_u \times r_v|}_{}\, dA \quad (|r_u \times r_v| = |\nabla f|)$$

Problem 8: Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle -x, 2y, -z\rangle$ & S is the portion of $y = 3x^2+3z^2$ that lies between $y=0$ & $y=6$. (oriented positively wrt y-axis)



- D: $3x^2+3z^2=6 \Rightarrow x^2+z^2 \le 2$

- $\vec{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{\langle 6x, -1, 6z\rangle}{|\nabla f|}$

- $\vec{n} = \dfrac{\langle -6x, 1, -6z\rangle}{|\nabla f|}$

(fix orientation)

$\vec{r}(x,z) = \langle x, 3x^2+3z^2, z\rangle$

$$\begin{vmatrix} & i & j & k \\ r_x = & 1, & 6x, & 0 \\ r_z = & 0, & 6z, & 1 \end{vmatrix} =$$

$i(6x) - j(1) + k(6x)$

$= \langle 6x, -1, 6x\rangle$

- $\vec{F}(x, 3x^2+3y^2, z)\cdot\vec{n} = \langle -x, 6x^2+6y^2, -z\rangle \cdot \dfrac{\langle -6x, 1, -6z\rangle}{|\nabla f|} = \dfrac{1}{|\nabla f|}(12(x^2+z^2))$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \frac{1}{|\nabla f|}\cdot(12(x^2+z^2))\,dS = \iint_D \frac{1}{\cancel{|\nabla f|}}\cdot(12(x^2+z^2))\,\underbrace{\cancel{|\nabla f|}\,dA}_{dS}$$

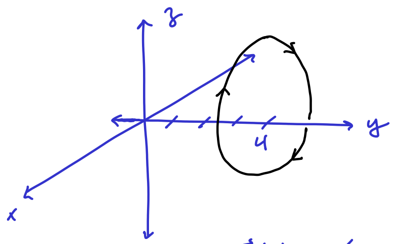
$$= \iint_D (12(x^2+z^2))\,dA \quad = \int_0^{2\pi}\int_0^{\sqrt{2}} 12\,r^2\cdot r\,dr\,d\theta$$

$x = r\cos\theta$
$z = r\sin\theta$

Flux: $\iint_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$

Stokes theorem: $\int_C \vec{F}\cdot d\vec{r} = \iint_S \boxed{\text{curl}\vec{F}\cdot d\vec{s}}$

Problem 9: Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = \langle -yz, (4y+1), xy\rangle$ and C is the circle of radius 3 at $y=4$ & perpendicular to the y-axis. C has a negative orientation wrt y.



- $\text{Curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yz & 4y+1 & xy \end{vmatrix} = \boxed{\langle x, -2y, z\rangle}$

- What surface should I use?

$\vec{r}(t) = \langle 3\cos t, 4, 3\sin t\rangle$

Find S such that $S \cap (y=6) = C$

$\boxed{y = x^2+z^2-5}$ $\qquad 4 = x^2+z^2-5 \Rightarrow x^2+z^2 = 9$

$\iint_S \langle x,-2y,z\rangle\cdot d\vec{s}$ $\qquad f(x,y,z) = x^2+z^2-y-5$

$\nabla f = \langle 2x,-1,2z\rangle$

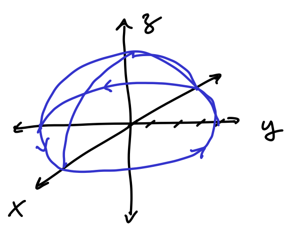
$\vec{F}(x, x^2+z^2-5, z) = \langle x, -2(x^2+z^2-5), z\rangle = \langle x, -2x-2z^2+10, z\rangle$

$\langle x, -2x^2-2z^2+10, z\rangle\cdot\langle 2x,-1,2z\rangle = 2x^2+2x^2+2z^2-10+2z^2$

$= 4x^2+4z^2-10 = h(x,z)$

$\int\int_D h(x,z)\,dA$ $\qquad$ where $D = \{(x,z) : x^2+z^2 \le 3\}$

Problem 10: Use Stokes theorem to evaluate $\iint_S \text{curl}\vec{F}\cdot d\vec{s}$ where $\vec{F} = \langle y, -x, yx^3\rangle$ & S is the portion of the sphere of radius 4, w/ $z \ge 0$ & upwards orientation.



$\oint_C \vec{F}\cdot d\vec{s}$

$\boxed{\vec{r}(t) = \langle 4\cos t, 4\sin t, 0\rangle}$ $\qquad \vec{r}\,'(t) = \langle -4\sin t, 4\cos t, 0\rangle$

$\int_C \vec{F}(r(t))\cdot\vec{r}\,'(t)\,dt$

$\vec{F}(\vec{r}(t)) = \langle 4\sin t, -4\cos t, 4^4\sin t\cos^3 t\rangle$

$\boxed{\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)} = \langle 4\sin t, -4\cos t, 4^4\sin t\cos^3 t\rangle\cdot\langle -4\sin t, 4\cos t, 0\rangle$

$= -16\sin^2 t - 16\cos^2 t = -16 \rightsquigarrow -16\int_0^{2\pi} dt = (2\pi)(-16)$

Summary

$\iint_S \nabla\times\vec{F}\cdot d\vec{s} = \int_C \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = -16$

we have:

$-16\int_0^{2\pi} dt = (-16)(2\pi)$