Thursday June 24, 2021
MTH 164 Lecture Notes

$g(x,y)$ $r(u,v)$

$r_u = \langle x_u, y_u, z_u \rangle$

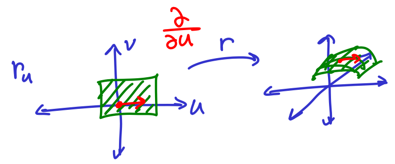
$\iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA$

$\iint_D f(r(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$

Section 16.7 (Continued): Surface Integrals of Vector Fields

- For a surface $z = g(x,y)$ given by the graph of g we form the unit normal vector:

$$\vec{n} = \frac{\langle -g_x, -g_y, 1\rangle}{|\langle -g_x, -g_y, 1\rangle|}$$



This determines an orientation on the surface. You can think of it as the "upward" orientation.

- For a parameterized surface $\vec{r}(u,v)$, we have, instead:

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

- Note that it's easy to see these are indeed perpendicular to any tangent vector of the surface.

Definition: If $\vec{F}$ is a continuous vector field defined over an oriented surface S with unit normal $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS.$$

This integral is also called the flux of $\vec{F}$ across S.

Let's find a more computationally useful formula.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,dS \quad \text{"curved"}$$

$$= \iint_D \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\cdot|\vec{r}_u\times\vec{r}_v|\,dA \quad \text{"flat"}$$

$$= \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

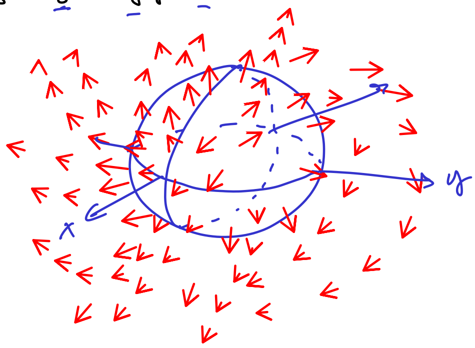
$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA}$$

When S is the graph of $g(x,y)$, & $\vec{F} = \langle P, Q, R\rangle$, we have:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-P g_x - Q g_y + R)\,dA}$$

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA}$$

Example: Find the flux of the vector field $\vec{F}(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere
$x^2+y^2+z^2=1$



- First, let's parameterize the sphere:

$$r(\theta,\varphi) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$$

- $\vec{r}_\varphi \times \vec{r}_\theta = \langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle$

- $\vec{F}(\vec{r}(\theta,\varphi)) = \langle \cos\varphi, \sin\varphi\sin\theta, \sin\varphi\cos\theta \rangle$

- $\vec{F}(\vec{r}(\theta,\varphi))\cdot(\vec{r}_\varphi \times \vec{r}_\theta) = \langle \cos\varphi, \sin\varphi\sin\theta, \sin\varphi\cos\theta \rangle \cdot \langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle$

$$= \sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\sin^2\theta + \sin^2\varphi\cos\varphi\cos\theta$$

What is a normal vector field normal to the sphere in cartesian coordinates?

$\langle \pm x^2, \pm y^2, z^2 \rangle$

$\langle x, y, z \rangle$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_\varphi \times \vec{r}_\theta)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(2\sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\sin^2\theta\right) d\varphi\, d\theta$$

$$= \int_0^{\pi} 2\sin^2\varphi\cos\varphi\, d\varphi \cdot \int_0^{2\pi}\cos\theta\, d\theta + \int_0^{2\pi}\sin^3\varphi\, d\varphi \cdot \int_0^{\pi}\sin^2\theta\, d\theta$$

$$= 0 + \int_0^{2\pi}\sin^3\varphi\, d\varphi \cdot \int_0^{\pi}\sin^2\theta\, d\theta$$

$$= \frac{4\pi}{3}$$

Applications:

1. If $\vec{E}$ is an electric field, $\iint_S \vec{E}\cdot d\vec{S}$ is the electric flux.

   Gauss' law says:

$$Q = \varepsilon_0 \iint_S \vec{E}\cdot d\vec{S} \quad = \text{net charge enclosed by } S.$$

2. Heat flow: $F = -K\nabla u$ (↑ conductivity; ↖ temperature)

   The rate of heat flow across a surface S is

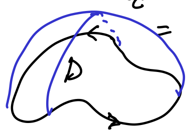
$$\iint_S \vec{F}\cdot d\vec{S} = -K\iint_S \nabla u \cdot d\vec{S}$$

Section 16.8 + 16.9 : Stokes' Theorem + Divergence Theorem

Stokes' Theorem: Let S be a closed, piecewise-smooth, oriented surface that is bounded by a simple, closed, piecewise-smooth, positively oriented curve C. Let $\vec{F}$ be a vector field whose components have continuous first-order partial derivatives on an open region $U \subset \mathbb{R}^3$ containing S. Then:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$

$$\oint_C F \cdot dr = \iint_D (\text{curl} F)$$



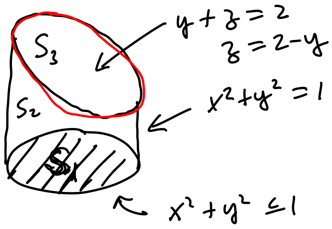
Challenge Question: What do Stokes' & Green's theorems have to do with integration by parts?

Divergence Theorem: Let E be a simple solid region with boundary S. Give S an outward orientation. Let $\vec{F}$ be a vector field whose components have continuous first-order partial derivatives on an open region containing E. Then:

$$\iint_S F \cdot dS = \iiint_E \text{div}\, \vec{F}\, dV$$

Examples:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$



Evaluate: $\int_C F \cdot dr$ where $F(x,y,z) = \langle -y^2, x, z^2 \rangle$ where C is the curve of intersection of $y + z = z$ & $x^2 + y^2 = 1$

1. Calculate the curl of F. $\vec{\nabla} \times \vec{F}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y^2 & x & z^2 \end{vmatrix} = i(0) - j(0) + k(1 + 2y)$$

$$= \langle 0, 0, 1 + 2y \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (1 + 2y)\, dA = \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta = \pi$$

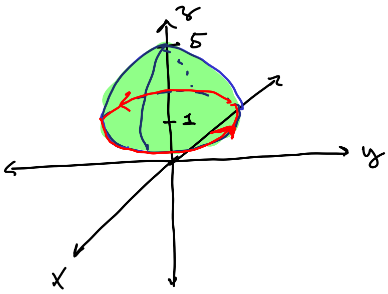
$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P g_x - Q g_y + R\right) dA$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\, dA$

Example: Use Stokes' theorem to evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where $F = \langle z^2, -3xy, x^3y^3\rangle$

S is the part of $z = 5 - x^2 - y^2$ above $z = 1$



$5 - x^2 - y^2 = 1$

$4 = x^2 + y^2$

$\vec{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle$

$\vec{F}\cdot d\vec{r} = \vec{F}(\vec{r})\cdot\vec{r}'\,dt$

$$\vec{F}(\vec{r}(t)) = \langle 1, -3(2\cos t)(2\sin t), \overset{8}{(2\cos t)^3}\cdot\overset{8}{(2\sin t)^3}\rangle$$

$$= \langle 1, -12\cos t\sin t, 64(\cos t\sin t)^3\rangle$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = \langle 1, -12\cos t\sin t, 64(\cos t\sin t)^3\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle$$

$$= -2\sin t - 24\cos^2 t\sin t$$

$$\int_0^{2\pi}(\overset{A}{-2\sin t} - \overset{B}{24\cos^2 t\sin t})\,dt = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} \quad \text{by Stokes'}$$

$$A = 2\cos t\Big|_0^{2\pi} = 2(1-1) = 0$$

$u = \cos t$

$du = -\sin t\,dt$

when $t = 0,$

$t = 2\pi$

$$24\int_1^1 u^2\,dt = 0$$

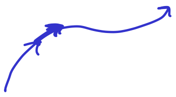
$\boxed{= 0}$

Summary of Chapter 16

Sections (16.1 – 16.3) : Vector fields, line integrals, fundamental theorem of line integrals.

- line integrals, For C given by $\vec{r}(t)$, & f a function we have (curve)

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt$$



$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}\,dt$$

$ds = |r'(t)|dt$

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|r'(t)|\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot r'(t)\,dt$$

$dS = |\vec{r}_u \times \vec{r}_v|\,dA$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\cdot|\vec{r}_u \times \vec{r}_v|\,dA$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(r(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

- FTLI : $\int_C \nabla f\cdot d\vec{r} = f(\vec{r}(a)) - f(\vec{r}(b))$

Sections 16.6 + 16.7 : Parametric surfaces, areas, + surface integrals

- We can write a surface as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

- Surface $A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA$

when S is given as the graph of $z = g(x,y)$ we have

$$A(S) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

- The integral of a function $f(x,y,z)$ over a surface S is

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\cdot|\vec{r}_u \times \vec{r}_v|\,dA$$

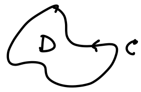
- The integral of a vector field $\vec{F}$ over a surface S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

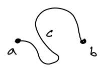
double integral ⟷ Big S   single integral ⟷ little s

Sections 16.4 + 16.5 : Green's Theorem + Curl + Divergence.

- Green's theorem: $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$



$f(\vec{r}(a)) - f(\vec{r}(b)) = \int_C \nabla f(\vec{r})\, \vec{r}'\, dt$



- Vector form: $\int_C \vec{F}\cdot d\vec{r} = \iint_D (\vec{\nabla}\times\vec{F})\cdot \vec{k}\, dA$

curl

$\langle \vec{i}, \vec{j}, \vec{k} \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

Sections 16.8 + 16.9.

- Stokes' Theorem: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

- Divergence Theorem: $\iint_S F\cdot dS = \iiint_E \text{div}\,\vec{F}\, dV$

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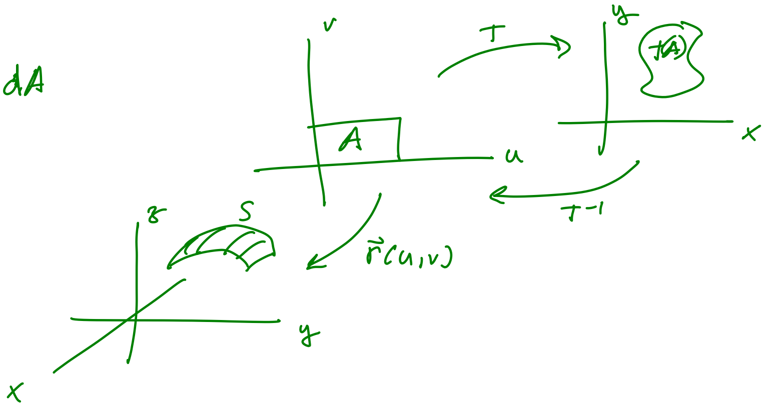
$dS = |\vec{r}_u \times \vec{r}_v|\, du\,dv$

$du\,dv = dA$

$d\vec{S} = (\vec{r}_u \times \vec{r}_v)\, du\,dv$



Stokes' Theorem: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

Example (Stokes' theorem): Evaluate $\int_C F\cdot dr$, $F = \langle z^2, y^2, x \rangle$

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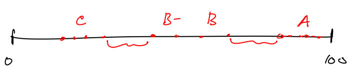
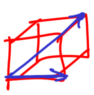
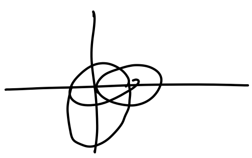


PartA : will replace lowest midterm grade.





$2\overline{)180}$ = 90