Wednesday, June 23, 2021
MTH 164 Lecture Notes

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Section 16.5: Curl and Divergence

$\vec{\nabla}$

- Curl

"How much is it swirling around the point (x,y,z)?"

The **curl** of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is defined by

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \operatorname{curl} \vec{F}$$

Example: $\vec{F}(x,y,z) = \langle \overset{P}{xz}, \overset{Q}{xyz}, \overset{R}{-y^2} \rangle$

$$\left.\begin{array}{ll} P_z = x & P_y = 0 \\ Q_x = yz & Q_z = xy \\ R_x = 0 & R_y = -2y \end{array}\right\} \quad \vec{\nabla} \times \vec{F} = (-2y - xy)\vec{i} + (x - 0)\vec{j} + (yz - 0)$$

$$= \langle -y(2+x), x, yz \rangle$$

Theorem: If a function of 3-variables f has continuous 2^{nd} order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \vec{0}$$

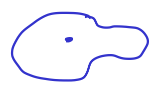
This follows from Clairaut's theorem:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\vec{\nabla} \times (\vec{\nabla} f) = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$$= \vec{0}$$

$$\left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$



In other words: If $\vec{F}$ is conservative, then $\operatorname{curl} \vec{F} = \vec{0}$

Roughly speaking, this is because $\vec{F}$ doesn't have "singularities" if it's conservative.

We also have the converse!

Theorem: If $\vec{F}$ is a vector field defined for all of $\mathbb{R}^3$ whose component functions have continuous 2^{nd} order partial derivatives, and $\operatorname{curl}\vec{F} = 0$, then $\vec{F}$ is conservative.

The proof of this theorem uses Stokes' theorem & will be proved later.

$$\boxed{\vec{\nabla}\times\vec{F} = \vec{0} \iff \vec{F} \text{ is conservative}}$$

Example: Show that $\vec{F}(x,y,z) = \langle \overset{P}{y^2z^3}, \overset{Q}{2xyz^3}, \overset{R}{3xy^2z^2} \rangle$ is conservative.

Then find f such that $\nabla f = \vec{F}$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

- $\frac{\partial P}{\partial y} = 2y\cdot z^3 \qquad \frac{\partial Q}{\partial x} = 2yz^3 \qquad \frac{\partial R}{\partial x} = 3y^2z^2$

  $\frac{\partial P}{\partial z} = 3y^2z^2 \qquad \frac{\partial Q}{\partial z} = 6xyz^2 \qquad \frac{\partial R}{\partial y} = 6xyz^2$

$$(6xyz - 6xyz^2)\vec{i} + (3y^2z^2 - 3y^2z^2)\vec{j} + (2yz^3 - 2y\cdot z^3)\vec{k} = \vec{0}$$

$\Rightarrow F$ is conservative.

- $f_x(x,y,z) = y^2z^3 \quad\Rightarrow\quad f(x,y,z) = y^2z^3x + K_1(y,z)$

  $f_y(x,y,z) = 2xyz^3 \quad\Rightarrow\quad f(x,y,z) = xy^2z^3 + K_2(x,z) \quad\Rightarrow\quad \boxed{f(x,y,z) = y^2z^3x + C}$

  $f_z(x,y,z) = 3xy^2z^2 \quad\Rightarrow\quad f(x,y,z) = xy^2\cdot z^3 + K_3(x,y)$

- Divergence

"How much fluid is coming out of or going into a point?"

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

Example: $\vec{F}(x,y,z) = \langle \overset{P}{xz}, \overset{Q}{xyz}, \overset{R}{-y^2} \rangle$

$$\vec{\nabla}\cdot\vec{F} = z + xz$$

Theorem: If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have a continuous second-order partial derivatives, then

$$\operatorname{div}\operatorname{curl} F = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$$

This follows from Clairaut again.

Vector forms of Green's Theorem

Note: $\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy \quad , F = \langle P, Q\rangle$

$\Rightarrow$ In $\mathbb{R}^2$, $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

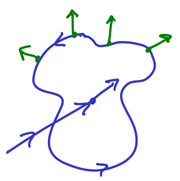
$\left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$

And $\text{curl}(F)\cdot\vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

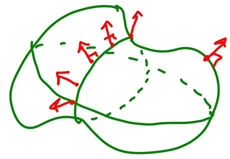
$\therefore$ Green's Theorem can be re-written: $\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\text{curl}\,\vec{F})\cdot\vec{k}\, dA}$

- If C is the trace of $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} \quad a \le t \le b$



unit tangent: $\boxed{\vec{T}(t) = \frac{1}{|\vec{r}'(t)|}\langle x'(t), y'(t)\rangle}$

unit normal: $\boxed{\vec{n}(t) = \frac{1}{|\vec{r}'(t)|}\langle y'(t), x'(t)\rangle}$



$\Rightarrow$ We can re-write Green's theorem:

$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$

$$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \text{div}\,\vec{F}(x,y)\,dA}$$

Back at 10:06

Since $\oint_C \vec{F}\cdot\vec{n}\,ds = \int_a^b (\vec{F}\cdot\vec{n})(t)\,|r'(t)|\,dt = \int_a^b \frac{1}{|r'(t)|}(Py' - Qx')\cdot|r'(t)|\,dt$

$= \int_a^b P\,dy - Q\,dx$

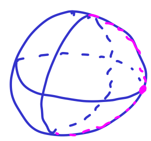
Section 16.6: Parametric surfaces and their areas

$r(\varphi, \theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$

$x = \sin\varphi\cos\theta$

$y = \sin\varphi\sin\theta$

$z =$



It's like a curve, but with two variables:

$$\boxed{\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}}$$

Examples

- $r(u,v) = \langle 2\cos u, v, 2\sin u\rangle$
- $r(u,v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u+\cos v\rangle$
- $r(u,v) = \langle u, v, \sin(u^2 - 2v)\rangle$

$f(x,y) = \sin(x^2 - 2y)$

- Tangent planes

Example: Find the plane tangent to $x = u^2$, $y = v^2$, $z = u + 2v$, at the point $(1,1,3)$

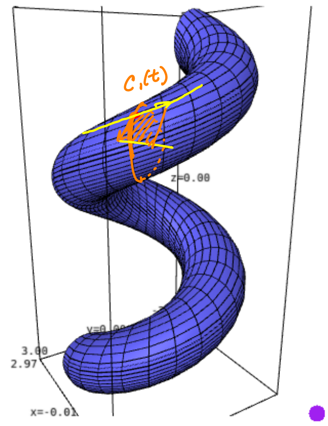
$\vec{r}_u = \langle x_u, y_u, z_u \rangle = \langle 2u, 0, 1 \rangle$ $\qquad \dfrac{\partial x}{\partial u} \quad \dfrac{\partial y}{\partial u} \quad \dfrac{\partial z}{\partial u}$

$dS = |\vec{r}_u \times \vec{r}_v|\, du\, dv$ $\qquad \vec{r}_v = \langle x_v, y_v, z_v \rangle = \langle 0, 2v, 2 \rangle$ $\qquad \dfrac{\partial x}{\partial v} \quad \dfrac{\partial y}{\partial v} \quad \dfrac{\partial z}{\partial v}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = \langle -2v, -4u, 4uv \rangle$$

at $(1,1,3)$, $u=1, v=1 \Rightarrow \vec{n} = \langle -2, -4, 4 \rangle$

$\Rightarrow$ The tangent plane is $-2(x-1) - 4(y-1) + 4(z-3) = 0$



Surface Area of parameterized surface

$r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, $(u,v) \in D$

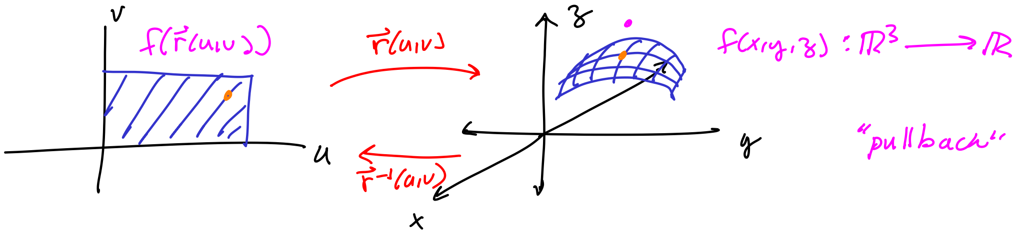
$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$$

where $r_u = \langle x_u, y_u, z_u \rangle$ $\qquad r_v = \langle x_v, y_v, z_v \rangle$

Section 16.7: Surface Integrals

- The surface integral of a function $f(x,y)$ over a surface S given by $\vec{r}(u,v)$ is

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v|\, dA}_{dS}$$



$f(\vec{r}(u,v)): \mathbb{R}^2 \longrightarrow \mathbb{R}$

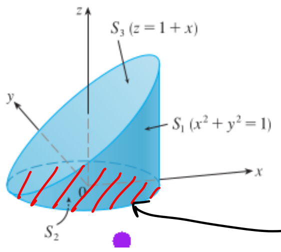
$r(u,v) = \langle u, v, g(u,v) \rangle$

$z = g(x,y) = \cancel{xy} + \cancel{yz} + \cancel{xz}$

$S = \Gamma(g)$

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \cdot \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

Example: Evaluate $\iint_S z\,dS$. S is the surface whose sides S_1 are given by a cylinder $x^2+y^2=1$, whose bottom is the $x^2+y^2\le 1$ in $z=0$ plane, whose top is given by $z=1+x$.



1. Parameterize S_1:

$S_1(\theta,z) = \langle x=\cos\theta,\ y=\sin\theta,\ z=z\rangle$ $0\le\theta\le 2\pi$, $0\le z\le 1+\cos\theta$

$S_2(x,y) = \langle r,\theta,0\rangle$, $0\le r\le 1$, $0\le\theta\le 2\pi$

$dS = |\vec r_u \times \vec r_v|\,du\,dv$

$$\vec r_\theta \times \vec r_z = \begin{vmatrix} \vec i & \vec j & k \\ \vec r_\theta = \langle -\sin\theta, & \cos\theta, & 0\rangle \\ \vec r_z = \langle 0, & 0, & 1\rangle \end{vmatrix} = \langle\cos\theta,\sin\theta,0\rangle$$

$$|\vec r_\theta \times \vec r_z| = 1$$

$$\iint_{S_1} z\,dS = \iint_D z|\vec r_\theta \times \vec r_z|\,d\theta\,dz = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{3\pi}{2}$$

2. Integrate over S_3: $z=1+x$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\cdot\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

$S_3(x,y) = \langle x, y, 1+x\rangle$

$$\vec r_x \times \vec r_y = \begin{vmatrix} i & j & k \\ r_x = \langle 1, & 0, & 1\rangle \\ r_y = \langle 0, & 1, & 0\rangle \end{vmatrix} = \langle 1, 0, 1\rangle$$

$|\vec r_x \times \vec r_y| = \sqrt{2}$

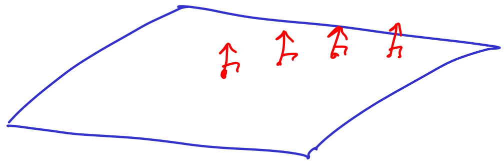
$\frac{\partial z}{\partial x} = 1$ $\frac{\partial z}{\partial y} = 0$

$$\int_0^{2\pi}\int_0^1 (1+x)\sqrt{2}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r+r^2\cos\theta)\sqrt{2}\,dr\,d\theta =$$

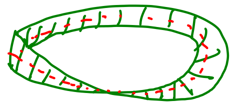
Solution: $\frac{3\pi}{2} + \left(\frac{3}{2}+\sqrt{2}\right)\pi$

$\boxed{\text{(The integral over } S_2 \text{ will be zero since } z=0 \text{ in } S_2)}$

Oriented Surfaces.



non-orientable



(möbius strip)

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$ unit normal vector field for $\vec{r}(u,v)$