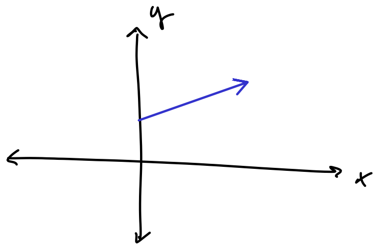
Tuesday, June 22, 2021
MTH 164 Lecture Notes



Short Review of line integrals

- A vector field is a map from $\mathbb{R}^2 \longrightarrow \{\text{vectors in } \mathbb{R}^2\}$

  Example: $\vec{F}(x,y) = \langle y^2, x+y \rangle$

- Some formulas:

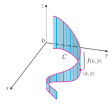
$f(\vec{r}(t))$ $\quad ds = |\vec{r}\,'(t)|\,dt$

3 $$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

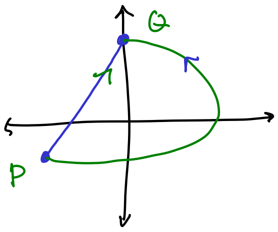
$$\boxed{\int_C f(\vec{x})\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$



- A line integral measures the "area of a curtain" under a curve, "size" of the graph of a function:



Example: Evaluate $\int_C y^2\,dx + x\,dy$ $\qquad \boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}}$

a) Where C is the line segment from $P(-5,-3)$ to $Q(0,2)$

b) where C is the arc of the parabola $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$.

Direction vector

$\vec{PQ} = \langle 5, 5 \rangle = v$

$r(t) = \langle 0,2 \rangle + t\langle 5,5 \rangle$

a) $\vec{r}(t) = \langle 5t-5,\ 5t-3 \rangle$ $\qquad \boxed{0 \le t \le 1}$

$\vec{r}\,'(t) = \langle 5, 5 \rangle$

$\Rightarrow$ we have $\int_0^1 (5t-3)^2 \cdot 5\,dt + \int_0^1 (5t-5)\cdot 5\,dt$

$= \int_0^1 (25t^2 - 25 + 4)\,dt = -\frac{5}{6}$

b) $\vec{r}(t) = \langle 4-t^2, t \rangle$ $\quad -3 \le t \le 2$ $\quad x = 4-y^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\quad y$

$\vec{r}\,'(t) = \langle -2t, 1 \rangle$

$$\int_{-3}^{2} t^2(-2t)\,dt + \int_{-3}^{2} (4-t^2)\,dt$$

$$= \int_{-3}^{2} (-2t^3 - t^2 + 4)\,dt = -\frac{2}{4}t^4 - \frac{1}{3}t^3 + 4t \Big|_{-3}^{2} = 40\tfrac{5}{6}$$

- We can also integrate vector fields using the dot product $\qquad F \cdot dr$

13 Definition Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

Example: Find the work done by $\vec{F} = \langle x^2, -xy \rangle$ in moving a particle along $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$

- $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle \cdot \vec{F}(\vec{r}(t)) = \cos^2 t - \cos t \sin t$

$$\int_0^{\pi/2} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{\pi/2} (-2\cos^2 t \sin t)\,dt = -\frac{2}{3}$$

Section 16.3: The fundamental theorem for line integrals

Recall the FTC: $\int_a^b F'(x)\,dx = F(b) - F(a)$

1 **Theorem:** Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or 3 variables whose gradient vector ∇f is continuous on C. Then

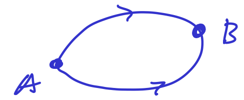
$$\int_C \boxed{\nabla f} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$\oint_C F \cdot dr = 0$

Proof: $\int_C \nabla f \cdot d\vec{r} = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$

(chain rule) (FTC)

$\int_b^a \sim dt$



- path indep.
- conservative

2 **Theorem:** $\int_C \vec{F} \cdot d\vec{r}$ is path independent in D iff $\oint_C \vec{F} \cdot d\vec{r} = 0$ $\forall$ closed paths C.

y n

3 **Theorem:** Suppose $\vec{F}$ is a vector field continuous on an open region D. If $\int_C \vec{F} \cdot d\vec{r}$ is path-independent, then $\vec{F}$ is conservative: $\exists$ a function f such that $\nabla f = \vec{F}$.

4 **Theorem:** Suppose $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is a conservative vector field, where P + Q have continuous first-order partial derivatives throughout D. We have:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Green's theorem

5 **Theorem:** The previous theorem is iff: $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative $\Longleftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Examples

1. Is $\vec{F}(x,y)$ conservative?

- $\vec{F}(x,y) = \langle x-y,\ x-2 \rangle = \underbrace{(x-y)}_{P(x,y)}\,\vec{i} + \underbrace{(x-2)}_{Q(x,y)}\,\vec{j}$

$$\frac{\partial Q}{\partial x} = 1 \neq \frac{\partial P}{\partial y} = -1 \implies \text{no, not conservative.}$$

- $\vec{F}(x,y) = \langle \underbrace{3+2xy}_{P},\ \underbrace{x^2 - 3y^2}_{Q} \rangle$

$$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} = 2x \implies \text{yes, conservative.}$$

WARNING: Don't try to apply FTLI to non-conservative v.f.

4. a) If $\vec{F}(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ find a function f such that $\vec{F} = \nabla f$

b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ for C given by $\vec{r}(t) = \langle e^t \sin(t), e^t\cos(t)\rangle$ $\quad 0 \le t \le \pi$

a) Find $f(x,y)$ such that $\nabla f = F$, $\nabla f = \langle f_x, f_y\rangle \Rightarrow f$ must satisfy:

$$f_x(x,y) = 3+2xy \qquad \Rightarrow \qquad f(x,y) = 3x + x^2y + C(y)$$

$$f_y(x,y) = x^2 - 3y^2 \qquad\qquad f(x,y) = x^2y - y^3 + K(x)$$

$$-3x + x^2y + C(y) = x^2y - y^3 + K(x)$$

$$\Rightarrow C(y) = -y^3 \qquad \Rightarrow \quad f(x,y) = x^2y - 3x - y^3$$

$$K(x) = -3x$$

b) $$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} \overset{FTLI}{=} f(\vec{r}(b)) - f(\vec{r}(a)) = f(0,-e^{\pi}) - f(0,1)$$

$$= (e^{\pi})^3 - 1$$

$\vec{r}(b) = (e^{\pi}\cdot 0 , e^{\pi}(-1))$

$\vec{r}(a) = (0,1)$

Theorem: Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or 3 variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \boxed{\nabla f}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

5. If $\vec{F}(x,y,z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z}\rangle$, find f such that $\nabla f = \vec{F}$.

$$f_x(x,y,z) = y^2 \qquad \Rightarrow \quad f(x,y,z) = y^2x + K(y,z)$$

$$f_y(x,y,z) = 2xy + e^{3z} \qquad f(x,y,z) = xy^2 + e^{3z}\cdot y + C(x,z)$$

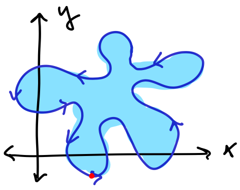
$$f_z(x,y,z) = 3ye^{3z} \qquad f(x,y,z) = ye^{3z} + D(x,y)$$

$$\Rightarrow f(x,y,z) = y^2x + e^{3z}\cdot y$$

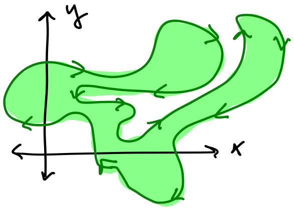
Break: Back at 10:06

Section 16.4: Green's Theorem!

First, some more topology: Orientation.



Positive

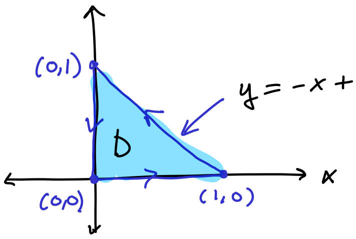


Negative

Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane + let D be the region bounded by C. If P + Q have continuous, first-order partial derivatives on an open region containing D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Example: Evaluate $\int_C \overset{P}{x^4}\,dx + \overset{Q}{xy}\,dy$ where C is the triangle w/ vertices $(0,0)$ $(1,0)$. $(0,1)$



$$\int_0^1 \int_0^{-x+1} y\,dy\,dx \qquad (-x+1)(-x+1) = x^2 - 2x + 1$$

$$= \int_0^1 \frac{1}{2} y^2 \Big|_0^{-x+1} dx = \int_0^1 \frac{1}{2}(-x+1)^2\,dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - 2x + 1)\,dx = \frac{1}{2} \cdot \frac{1}{3} - \frac{1}{2} + \frac{1}{2} = \frac{1}{6}$$

Example: Evaluate $\oint_C \overset{P}{(3y - e^{\sin x})}\,dx + \overset{Q}{(7x + \sqrt{y^4 - 1})}\,dy$ where C is $x^2 + y^2 = 9$

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\iint_D (7-3)\,dA = 4 \underline{\iint_D dA} = 4 \cdot \pi \cdot 9 = 36\pi$$

We can use Green's theorem to find area!

Green's theorem.

$$\text{Area}(D) = \iint_D 1\, dA = \iint \left(\frac{\partial}{\partial x}\Big(\underbrace{\tfrac{1}{2}x}_{Q}\Big) - \frac{\partial}{\partial y}\Big(\underbrace{-\tfrac{1}{2}y}_{P}\Big) \right) dx\,dy = \frac{1}{2} \oint_C (\overset{Q}{x}\,dy - \overset{P}{y}\,dx)$$

Example: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\pi r_1 r_2$

$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$, $0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -a\sin(t), b\cos(t) \rangle$

Green's theorem.

$$\iint_{D = \text{ellipse}} dA = \frac{1}{2} \int_0^{2\pi} a\cos(t) \cdot b\cos(t)\, dt + \frac{1}{2} \int_0^{2\pi} b\sin(t)\, a\sin(t)\, dt$$

$$= \frac{1}{2}ab \int_0^{2\pi} (\cos^2(t) + \sin^2(t))\, dt = \frac{1}{2}ab\,(t)_0^{2\pi} = a \cdot b \cdot \pi$$

Extended Green's theorem

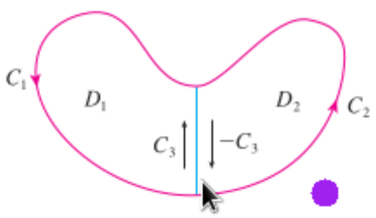


FIGURE 6

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_1 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\Rightarrow \int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

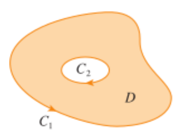


FIGURE 9

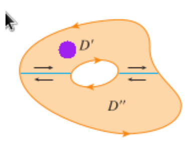


FIGURE 10

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA =$$

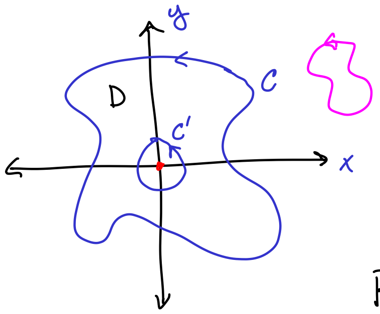
$$\iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,d$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

Example: If $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$ (P, Q), show that $\boxed{\int_C \vec{F}\cdot d\vec{r} = 2\pi}$

for every positively oriented, simple, closed curve enclosing the origin.



C'

$F\cdot dr = \langle P, Q\rangle \cdot \langle dx, dy\rangle = Pdx + Qdy$

$$\int_C (Pdx + Qdy) + \int_{-C'} (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$dr = \langle dx, dy\rangle$

$$P = \frac{-y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = 0$$

$$\int_C (Pdx + Qdy) = \int_{C'} (Pdx + Qdy)$$

$$\int_{C'} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\, dt$$

$\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t)\rangle$

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

$$\vec{F}(\vec{r}(t)) = \langle -\sin(t) + \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle = \sin^2(t) + \cos^2 t$$

$$\therefore \int_{C'} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} dt = 2\pi$$

when $x=0$, $(0, y, \frac{4}{y})$
0

$\boxed{xy + xz + zy = 4}$

$d^2 = x^2 + y^2 + z^2$

$2x = \lambda(y+z)$ $\qquad z = \frac{4}{y}$

$2y = \lambda(x+z)$

$2z = \lambda(x+y)$ $\qquad \boxed{\Rightarrow x = y = z}$

$\Rightarrow xy + xz + zy = 3x^2$

$xy + xz + zy = 4$

$z = y = x = \sqrt{\frac{4}{3}}$

$$d^2\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 3\frac{4}{\sqrt{3}} = 4$$

$\boxed{d = 2}$