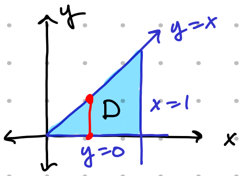
Thursday, June 17, 2021
MTH 164 Lecture Notes

Review: Sections 15.1 – 15.8

1. Evaluate

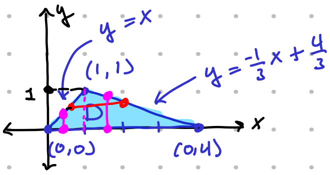
- $\iint_D x\,dA$ where D is enclosed by $y=x$, $y=0$, $x=1$



$$\int_0^1\int_0^x x\,dy\,dx = \int_0^1 xy\Big|_0^x dx = \int_0^1 x^2\,dx$$

$$= \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

- $\iint_D y\,dA$ where D is a triangle with vertices $(0,0)$, $(1,1)$ and $(4,0)$



$$\int_0^1\int_y^{-3y+4} y\,dx\,dy = \int_0^1 y\left(-3y+4-y\right)dy$$

$y - \frac{4}{3} = -\frac{1}{3}x$

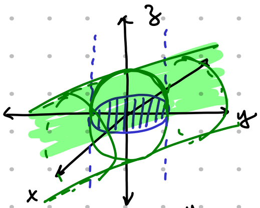
$$= \int_0^1(-4y^2+4y)\,dy = \frac{-4}{3}y^3 + \frac{4}{2}y^2\Big|_0^1$$

$-3y+4 = x$

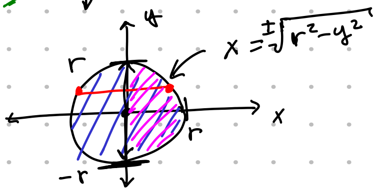
$$= \frac{-4}{3}+2$$

2. Find the volume

- Enclosed by the cylinders $x^2+y^2=r^2$ and $y^2+z^2=r^2 \rightsquigarrow z=\sqrt{r^2-y^2}$



$$2\int\int_D \sqrt{r^2-y^2}\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le r^2\}$$



$$8\int_0^r\int_0^{+\sqrt{r^2-y^2}}\sqrt{r^2-y^2}\,dx\,dy$$

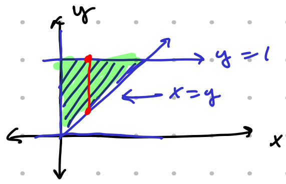
$$= 8\int_0^r\left(\sqrt{r^2-y^2}\right)\left(\sqrt{r^2-y^2}\right)dy$$

$$= 8\int_0^r(r^2-y^2)\,dy = 8\left(r^3-\frac{1}{3}r^3\right) = \boxed{\frac{8\cdot 2}{3}r^3}$$

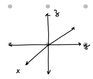
$$8\int_0^r\int_0^{\sqrt{r^2-y^2}}\int_0^{\sqrt{r^2-y^2}} dz\,dx\,dy$$

3. Change the order of integration + evaluate if $f(x,y)$ is given.

- $\int_0^1 \int_0^y f(x,y)\,dx\,dy$



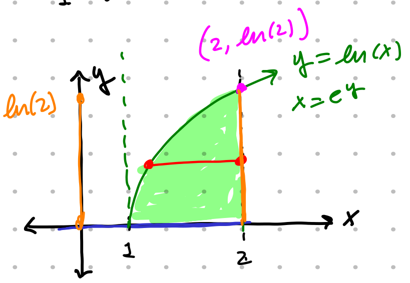
$$\int_0^1 \int_x^1 f(x,y)\,dy\,dx$$



- $\int_1^2 \int_0^{\ln(x)} f(x,y)\,dy\,dx$

$y = \ln(x)$

$e^y = x$



$$\int_0^{\ln(2)} \int_{e^y}^{2} f(x,y)\,dx\,dy$$

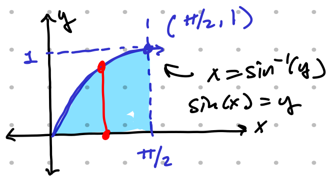
- $\int_0^1 \int_{\arcsin(y)}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\,dx\,dy$

$\arcsin(y)$

$x = \sin^{-1}(y)$

$x = \pi/2$

$\sin^{-1}(y) = \pi/2 \implies y = \sin(\pi/2) = 1$



$$\int_0^{\pi/2} \int_0^{\sin(x)} \cos x \sqrt{1+\cos^2 x}\,dy\,dx$$

$$\int_0^{\pi/2} \cos x \sqrt{1+\cos^2 x} \cdot \sin(x)\,dx$$

$$\int_0^{\pi/2} \sin x \cdot \cos x \sqrt{1+\cos^2 x}\,dx$$

$\frac{d}{dx} \frac{-1}{2} \cdot \frac{2}{3} \left(1+\cos^2 x\right)^{3/2}$ $\qquad \frac{-\cancel{2}}{\cancel{2}\cdot\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} \left(1+\cos^2 x\right)^{1/2} \left(\cancel{2}\cos x\,(-\sin x)\right)$

$$\boxed{\frac{-1}{2} \cdot \frac{2}{3} \left(1+\cos^2 x\right)^{3/2} \Big|_0^{\pi/2}}$$

4. Evaluate

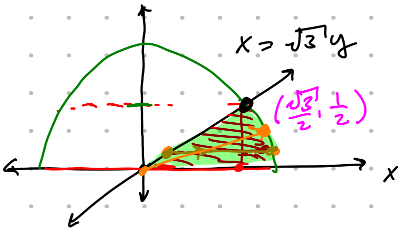
- $\int_0^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 \, dx\,dy$

$x^2 = 1-y^2$

$3y^2 = 1-y^2$

$4y^2 = 1 \quad y = \frac{1}{2}$

$\int_0^{\pi/6}\int_0^1 r^3\cos\theta\sin^2\theta \, dr\,d\theta$



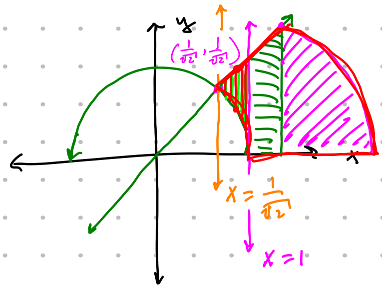
$x = r\cos\theta$
$y = r\sin\theta$

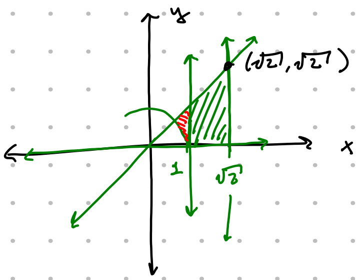
$u = \sin\theta \qquad du = \cos\theta\, d\theta$

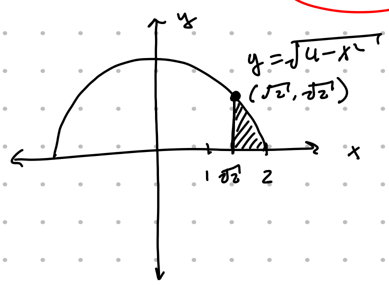
$$\int_0^{\pi/6}\int_0^1 r^3\cos\theta\sin^2\theta\, dr\,d\theta = \int_0^{\pi/6}\cos\theta\sin^2\theta\cdot\frac{1}{4}\, d\theta = \frac{1}{4}\int_0^{1/2}u^2\,du = \frac{1}{12}\left(\frac{1}{2}\right)^3$$

Break: Back at 10:05

- $\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$

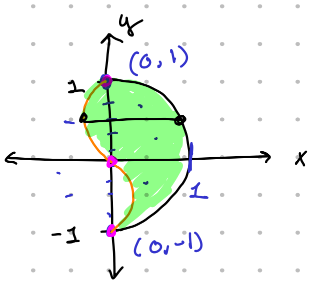






$$\boxed{= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\, dr\,d\theta}$$

- $\iint_D y\,dA$ where D is bounded by $x = y - y^3$ and $x = \sqrt{1-y^2}$



$$\int_{-1}^{1}\int_{y-y^3}^{\sqrt{1-y^2}} y\,dx\,dy$$

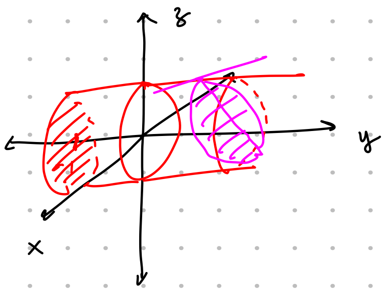
5. Find the Volume

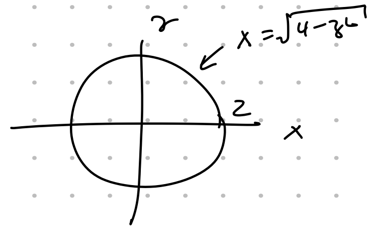
- Solid enclosed by $x^2+z^2=4$, $y=-1$, $y+z=4$

$y = 4 - z$

$z = 4 - y$

$x = r\cos\theta$
$z = r\sin\theta$



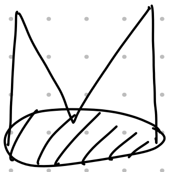


$$\int_0^{2\pi}\int_0^2\int_{-1}^{4-r\sin\theta} r\,dy\,dr\,d\theta$$

$$4\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{+\sqrt{4-z^2}}\int_{-1}^{4-z} dy\,dx\,dz$$

6. Evaluate

- $\iiint_E x^2\,dV$ where E is the solid within the cylinder $x^2+y^2=1$, above $z=0$ and below the cone $z^2=4x^2+4y^2$



$$\boxed{\int_0^{2\pi}\int_0^{1}\int_0^{2\cdot r} r\,dz\,dr\,d\theta}$$

$z=\sqrt{4x^2+4y^2}$

$x=r\cos\theta$

$y=r\sin\theta$

$\sqrt{4\cdot r^2}=2r$

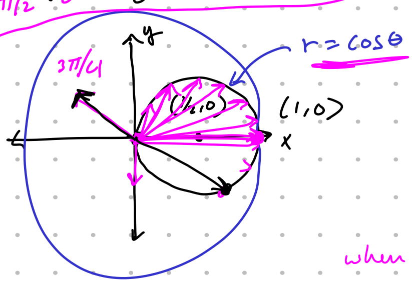
- $\iiint_E x^2\,dV$ where E is the solid within the cylinder $\boxed{x^2+y^2=x}$, above $z=0$ + below $z=-x^2-y^2+1$

$$\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_{0}^{-r^2+1} r^2\cos^2\theta\, r\,dz\,dr\,d\theta$$

Intersection of paraboloid + xy-plane

$0=-x^2-y^2+1$

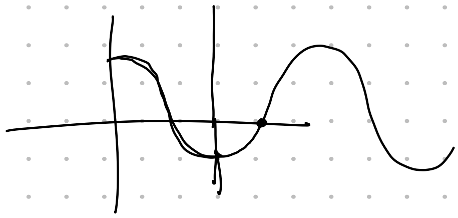
$\Rightarrow x^2+y^2=1$



$r^2=r\cos\theta$

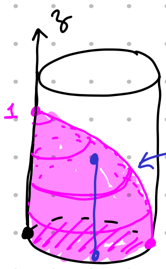
$r=\cos\theta$

$=-\frac{\sqrt{2}}{2}$



when $\theta=-\pi/2$

$\theta=-\pi/4$



$z=1-x^2-y^2$

$z(r)=1-r^2$

$z^2+x^2+y^2=a^2$ $\qquad$ $z(x,y)=\sqrt{a^2-x^2-y^2}$

- $$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \boxed{(x^2z+y^2z+z^3)}\,dz\,dx\,dy$$

$x=\rho\sin\varphi\cos\theta$

$y=\rho\sin\varphi\sin\theta$

$z=\rho\cos\varphi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} f(\rho,\theta,\varphi)\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Section 16.2: Line Integrals (Continued)

- Recall yesterday, we learned to calculate

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \underbrace{\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt}_{ds}$$

For curves $C(t) = \langle x(t), y(t)\rangle$ in $\mathbb{R}^2$

- Now, suppose $C(t) = \langle x(t), y(t), z(t)\rangle$ in $\mathbb{R}^3$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t)) \sqrt{(dx/dt)^2+(dy/dt)^2\ (dz/dt)^2}\,dt$$

We can also integrate vector fields along curves!

Definition: Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by $\vec{r}(t)$, $a \le t \le b$. Then the line integral of $\vec{F}$ along C is:

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \underbrace{\vec{r}\,'(t)\,dt}_{d\vec{r}} = \int_a^b \vec{F}\cdot\vec{T}\,ds$$

unit tangent.

$$\vec{F}(x,y) = \langle x(t), y(t)\rangle$$

$$\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$$

$$d\vec{r} = \vec{r}\,'(t)\cdot dt = \langle -\sin(t), \cos(t)\rangle\,dt$$

$$\vec{F}\cdot d\vec{r} = \left(-x(t)\cdot\sin(t) + y(t)\cos(t)\right)dt$$

$$\int_C \vec{F}\,d\vec{r} = \int_0^{2\pi} \left(-x(t)\cdot\sin(t) + y(t)\cos(t)\right)dt$$

Example: Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = \overset{\langle xy,\, yz,\, zx\rangle}{xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}}$ + C is given by $\vec{r}(t) = \langle t, t^2, t^3\rangle$ $(t\in[0,1])$

$$d\vec{r} = \vec{r}\,'(t)\,dt = \langle 1, 2t, 3t^2\rangle dt$$

$$\vec{F}\cdot d\vec{r} = \langle xy, yz, zx\rangle\cdot\langle 1, 2t, 3t^2\rangle dt = (xy + yz\cdot 2t + zx\cdot 3t^2)\,dt$$

We have $\displaystyle\int_0^1 (t^3 + 2t^6 + 3t^6)\,dt$

$$= \boxed{\frac{1}{4} + \frac{2}{7} + \frac{3}{7}}$$