Wednesday, June 16, 2021

MTH 164 Lecture Notes

$x^2+2y^2+3z^2=1$

Review: 14.3-14.8

1. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ $\qquad \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$

- $x^2+2y^2+3z^2=1$

$$F(x,y,z) = x^2+2y^2+3z^2-1$$
$$F_x = 2x \;,\; F_y = 4y \;,\; F_z = 6z$$
$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-2x}{6z} = \frac{-x}{3z} \;,\quad \frac{\partial z}{\partial y} = \frac{-4y}{6z} = \frac{-2y}{3z}$$

2. Find all points at which the direction of fastest change of the function $f(x,y) = x^2+y^2-2x-4y$ is $\vec{v} = \langle 1,1 \rangle$.

$\nabla f = \langle 2x-2, 2y-4 \rangle$ will be parallel to $\vec{v} \iff 2x-2 = 2y-4$

$\Rightarrow x-1 = y-2$

$\Rightarrow x = y-1$, Thus the points where the direction of fastest change is $\langle 1,1 \rangle$ are all the points on the line $x=y-1$.

3. Find the local min and max values and the saddle points

- $f(x,y) = x^2+xy+y^2+y$

1. Calculate Crit. points.

$$\left.\begin{aligned} f_x &= 2x+y = 0 \\ f_y &= x+2y+1 = 0 \end{aligned}\right\}$$

$2x+y = x+2y+1$

$x = y+1 \Rightarrow 2y+2+y = 0$

$3y = -2 \qquad y = -\frac{2}{3}$

$x - \frac{4}{3} + \frac{3}{3} = x - \frac{1}{3} = 0 \qquad x = \frac{1}{3}$

The only critical point is $\boxed{\left(\frac{1}{3}, -\frac{2}{3}\right)}$ $\qquad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$

$f_{xx} = 2 > 0$, $f_{yy} = 2$ $\quad f_{xy} = 1 \Rightarrow D = 4-1 = 3 > 0 \Rightarrow \left(\frac{1}{3}, -\frac{2}{3}\right)$ is a minimum.

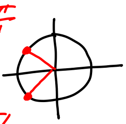
The minimum value of f is therefore $f\left(\frac{1}{3}, -\frac{2}{3}\right) = \frac{1}{9} - \frac{2}{9} + \frac{4}{9} - \frac{6}{9} = \frac{1-2+4-6}{9} = -\frac{1}{3}$

- $f(x,y) = e^x\cos(y)$

$$\left.\begin{aligned} f_x &= e^x\cos(y) = 0 \\ f_y &= -e^x\sin(y) = 0 \end{aligned}\right\}$$

$e^x\cos(y) = -e^x\sin(y) \Rightarrow \cos(y) = -\sin(y)$

$\Rightarrow y = \frac{4\pi n - \pi}{4}$

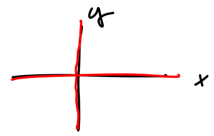


But of course, $\cos\left(\frac{4\pi n-1}{4}\right) \neq 0 \Rightarrow$ no critical points

- $f(x,y) = xy + e^{-xy}$

$$f_x = y - y e^{-xy} = 0$$
$$f_y = x - x e^{-xy} = 0$$

Critical points are $\{(x,y) : \text{either } x \text{ or } y = 0\}$



$$f_{xx} = y^2 e^{-xy}$$
$$f_{yy} = x^2 e^{-xy}$$
$$f_{xy} = 1 + yx e^{-xy} - e^{-xy}$$

$D(0,0) = 0$ $\Rightarrow$ 2nd derivative test fails!

Let $u = xy$

$$f(x,y) = f(u) = u + e^{-u}$$

$$\frac{d}{du} f(u) = 1 - e^{-u}$$

$$\frac{d^2}{du^2} f(u) = e^{-u} > 0$$

$\therefore$ x & y - axes are local mins.

$f_x = y - ye^{-xy} = 0$

$f_y = x - xe^{-xy} = 0$

$\Rightarrow$ when $y \neq 0$, $x = 0$: $1 = e^{-xy}$ | when $x \neq 0$, $y = 0$: $1 = e^{-xy}$

Critical points $\{(x,0), (0,y)\}$

$f_{xx} = y^2 e^{-xy}$ $\quad f_{xy} = 1 + xye^{-xy} - e^{-xy}$

$f_{yy} = x^2 e^{-xy}$

$\Rightarrow D(x,0) = D(0,y) = 0$

$\Rightarrow$ The 2^{nd} derivative test fails.

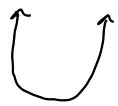
Do a substitution: Let $u = xy$. then $f(x,y) = f(u) = u + e^{-u}$

$\frac{d}{du} f(u) = 1 - e^{-u} = 0 \Rightarrow u = 0$

$\frac{d^2}{du^2} f(u) = e^{u} > 0 \Rightarrow u = 0$ is a local min.

$\Rightarrow$ All the critical points are also local minimums.

The minimum values are $f(x,0) = f(0,y) = 1$

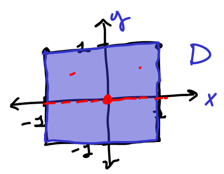


when $y = 0$, $f(x,y) = 2x^4$

4. Find a local minimum of $f(x,y) = 2x^4 + y^2 - xy^3$. Prove that it is a local min.

$f_x = 8x^3 - y^3$
$f_y = 2y - 3xy^2$

Clearly these are satisfied by the point $(0,0)$



$\left.\begin{array}{l} f_{xx} = 24x^2 \\ f_{yy} = 2 - 6xy \\ f_{xy} = -3y^2 \end{array}\right\}$ $D(0,0) = 0 \Rightarrow 2^{nd}$ derivative test fails!

"measure zero set"

- We have: $f(0,0) = 0$

Define $D = \{(x,y) : |x| \leq 1$ and $|y| \leq 1\}$

- Claim: $f(x,y) \geq 0 \quad \forall (x,y) \in D$

Note that $f(x,y) = 2x^4 + y^2 - xy^3 \geq y^2 - xy^3$ since $2x^4$ is always positive.

$\Rightarrow$ It suffices to show that $y^2 - xy^3 \geq 0 \quad \forall (x,y) \in D$.

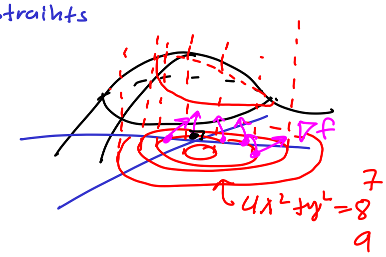
We have: $\quad y^2 - xy^3 \geq 0$

$\Leftrightarrow 1 - xy \geq 0$

$\Leftrightarrow 1 \geq xy$

But but since $|x| < 1$ & $|y| < 1$, $|xy| < 1$ so this is always true in D. $\square$

5. Find the extreme values of $f(x,y)$ subject to the constraint or constraints



- $f(x,y) = xy$; $4x^2+y^2=8$

$\boxed{\nabla f = \lambda \nabla g}$

$y = \lambda 8x$

$x = \lambda 2y$

$\boxed{4x^2+y^2=8}$

Note that niether x nor y can be zero.

$\frac{y}{4x} = 2\lambda = \frac{x}{y} \Rightarrow y^2 = 4x^2$

$\Rightarrow 2y^2 = 8 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2 \Rightarrow 4 = 4x^2 \Rightarrow x = \pm 1$

Critical points $(\pm 1, \pm 2)$

$f(1,2) = f(-1,-2) = 2 \leftarrow$ max $\qquad f(-1,2) = f(1,-2) = -2 \leftarrow$ min

$g(x,y,z) = x^4+y^4+z^4$

- $f(x,y,z) = x^2+y^2+z^2$; $x^4+y^4+z^4=1$

* by symmetry, we don't need to check $y=0$, $x \,\&\, z \neq 0$

$\nabla f = \langle 2x, 2y, 2z \rangle \qquad \nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle$

$\nabla f = \lambda \nabla g$

$\cancel{2x = 4\lambda x^3}$

$2y = 4\lambda y^3$

$2z = 4\lambda z^3$

$\cancel{x^4} + y^4 + z^4 = 1$

Case 1 : $x, y, \& z$ are all non-zero.

$\frac{2}{4\lambda} = x^2 = y^2 = z^2 \qquad x = \pm y = \pm z$

$3 \cdot x^4 = 1 \Rightarrow x^4 = \frac{1}{3} \Rightarrow \boxed{x = \pm 3^{-1/4} = y = z}$

Case 2 : $x=0$, $y \,\&\, z$ are non zero.

$y = \pm z \qquad 2y^4 = 1 \Leftrightarrow \boxed{y = z = \pm 2^{-1/4}}$

Case 3 : $x = y = 0$, $z \neq 0 \Leftrightarrow \boxed{z = \pm 1}$

$f(3^{-1/4}, 3^{-1/4}, 3^{-1/4}) = 3 \cdot (3^{-2/4}) = 3 \cdot 3^{-1/2} = \sqrt{3}$ ← max

$f(0, 2^{-1/4}, 2^{-1/4}) = 2(2^{-1/2}) = \sqrt{2}$

$f(0,0,1) = 1$ ← min.

$g(x,y) = x^2+z^2 \qquad h(x,y) = x+y$

- $f(x,y,z) = x+y+z$; $x^2+z^2=2$; $x+y=1$

$\nabla f = \langle 1,1,1 \rangle \qquad 1 = 2\lambda x + \mu$

$\nabla g = \langle 2x, 0, 2z \rangle \qquad 1 = \mu \qquad \Rightarrow 2\lambda x = 0 \Rightarrow x = 0,\ y = 1,\ z = \pm\sqrt{2}$

$\nabla h = \langle 1, 1, 0 \rangle \qquad 1 = 2\lambda z$

Critical points: $(0,1,\sqrt{2})$ & $(0,1,-\sqrt{2})$

$f(0,1,\sqrt{2}) = 1+\sqrt{2} \leftarrow$ max

$f(0,1,-\sqrt{2}) = 1-\sqrt{2} \leftarrow$ min

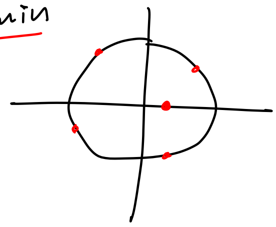
6. Find the extreme values of f on the region described by the inequality.

- $f(x,y) = 2x^2+3y^2-4x-5$ $\qquad$ $\underline{x^2+y^2 \le 16}$

Step 1: Check crit. points of $f(x,y)$. (Check interior)

$f_x = 4x-4 = 0 \Rightarrow x=1$ $\qquad$ $f(1,0) = 2-4-5 = 2-9 = -7$ ← global min

$f_y = 6y = 0 \Rightarrow y=0$



Step 2: Check the boundary. constraint: $\underline{x^2+y^2=16}$ ← $g(x,y)$

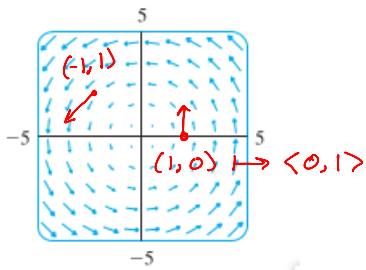
* Boundary	
$4x-4 = 2\lambda x$	when $y=0$, $x=\pm 4$
$6y = 2\lambda y$	when $y \neq 0$, $\frac{4x-4}{x} = 6$ $\quad 4x-4=6x \Rightarrow -4 = 6x-4x = 2x$
$x^2+y^2=16$	$\Rightarrow x=-2 \Rightarrow 4+y^2=16 \Rightarrow y = \pm\sqrt{12}$

$f(\pm 4, 0) = 2\cdot 8 \pm 16 - 5 = 27$ or -5 ← local min on the boundry ✗

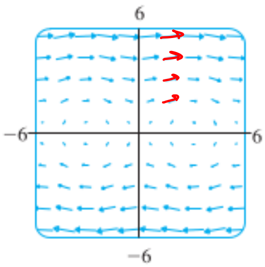
$f(-2, \pm\sqrt{12}) = 2\cdot 4 + 3\cdot 12 + 8 - 5 = 47$ ← global max.

Section 16.1: Vector Fields

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad$ $F(x,y): \mathbb{R}^2 \to \mathbb{R}$

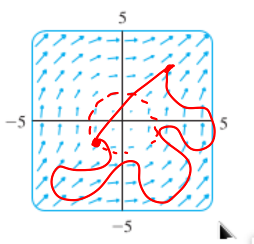


$\vec{F}(x,y) = \langle -y, x\rangle$



f

$\vec{F}(x,y) = \langle y, \sin(x)\rangle$



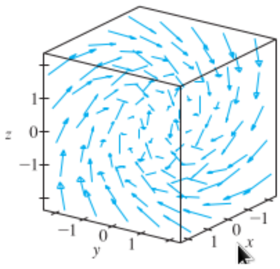
$\ln(x) = a$

$x = e^a$

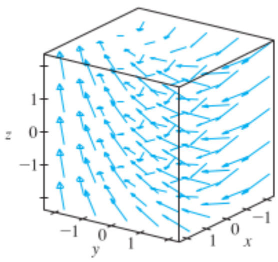
$\ln(1) = a$

$1 = e^a$

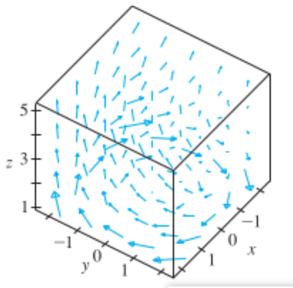
$\vec{F}(x,y) = \langle \ln(1+y^2), \ln(1+x^2)\rangle$



$\vec{F}(x,y,z) = \langle y, z, x\rangle$

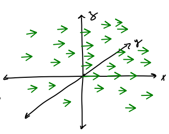


$\vec{F}(x,y,z) = \langle y, -2, x\rangle$

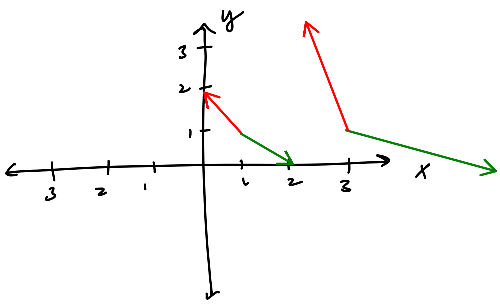


$\vec{F}(x,y,z) = \langle y/z, -x/z, z/4\rangle$

Sketch:



- $\vec{F}(x,y) = \langle 1,0,0\rangle$
- $\vec{F}(x,y) = \langle y, 0\rangle$
- $\vec{F}(x,y) = \langle y, y+2\rangle$
- $\vec{F}(x,y) = \langle x, -y\rangle$
- $\vec{F}(x,y) = \langle -y, x\rangle$



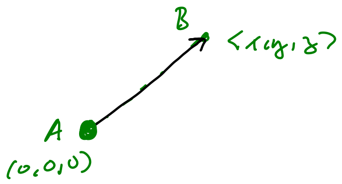
Examples:

Newton's Law of Gravitation: Force between two objects of mass m, M is

$$|\vec{F}| = \frac{m \cdot M \cdot G}{r^2}$$

gravitational constant

distance between



$$\vec{F}(\vec{x}) = \frac{mMG}{|\vec{x}|^2} \cdot \frac{\vec{x}}{|\vec{x}|} = -\frac{mMG}{|\vec{x}|^3} \cdot \vec{x}$$

Example: Suppose a charge Q is located at $(0,0,0)$. Coulomb's Law

$$\Rightarrow \qquad \vec{F}(\vec{x}) = \frac{\varepsilon q Q}{|\vec{x}|^3} \vec{x}$$

Force exerted by Q on q located at (x,y,z). $\quad \vec{x} = \langle x,y,z \rangle$

charge

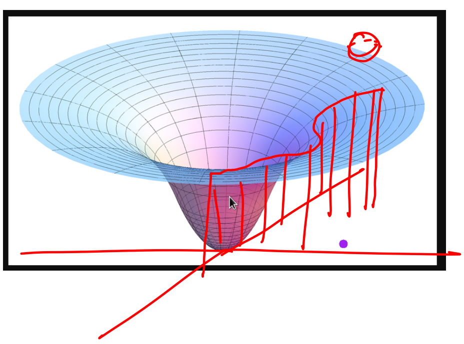
▫ Gradient Fields.

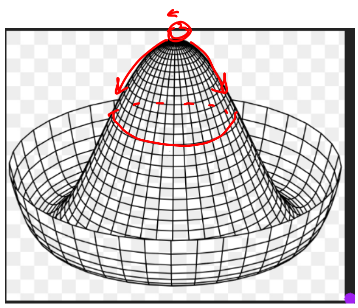
$f(x,y) = x^2 y - y^3$

$\nabla f = \langle 2xy, \; x^2 - 3y^2 \rangle$

Definition: A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some scalar function f.

Definition: If $\vec{F}$ is conservative + f satisfies $\nabla f = \vec{F}$, then f is called a potential function.

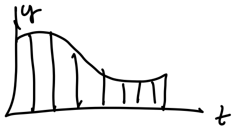




"Mexican hat"

Higgs boson.

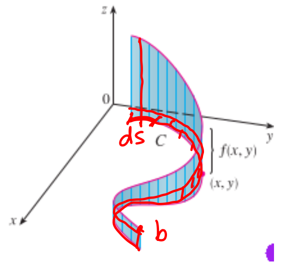
Section 16.2 : Line Integrals



y

$\vec{F}\cdot dr$

$f(x,y)$



$$\boxed{\int_C f(x,y)\,ds = \int_a^b \underbrace{f(x(t),y(t))}_{}^{1} \cdot \overbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}^{ds}\,dt}$$

$\vec{r}(t) = C(t) = \langle x(t), y(t)\rangle$

Example: Evaluate $\int_C (2 + x^2y)\,ds$ where C is the upper half of the unit circle.

$C(t) = \langle \cos(t), \sin(t)\rangle \quad , \quad 0 \le t \le \pi$

$f(x,y) = 2 + x^2y$

$\Rightarrow f(t) = 2 + \cos^2 t \cdot \sin(t)$

$\frac{dx}{dt} = -\sin(t) \qquad \frac{dy}{dt} = \cos(t)$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1$

Our integral becomes

$$\int_0^{\pi} \left(2 + \cos^2 t \cdot \sin(t)\right)\cdot dt$$

$u = \cos t \qquad du = -\sin(t)\,dt$

$$2\pi - \int_0^1 u^2\,du = 2\pi - \frac{1}{3}u^3\Big|_0^1$$

$$\boxed{= 2\pi - \frac{1}{3}}$$