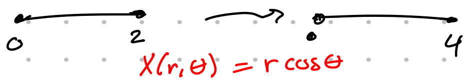
Tuesday, June 15, 2021
MTH 164 Lecture Notes

$\int_0^2 \sin(x^2)\, x\, dx$ $\quad \boxed{u = x^2}$ $\quad \frac{1}{2} du = x\, dx$




$x(r,\theta) = r\cos\theta$
$y(r,\theta) = r\sin\theta$

$x(\rho,\varphi,\theta)$ $\quad 0 \le \varphi \le \pi$

## Section 15.9: Change of Variables in multiple coordinates

In 1-dimensional calculus, we often did a "u-sub"

$x = g(u)$, $a = g(c)$ $\quad b = g(d)$

$$\int_a^b f(x)\, dx = \int_c^d f(x(u)) \boxed{\frac{dx}{du}}\, du$$

Yesterday, we learned about a couple changes of variables: cylindrical + spherical coordinates

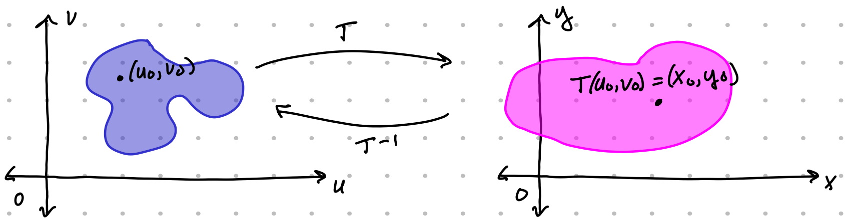
Today, we will consider general $C^1$-transformations between the uv-plane and the xy-plane.

$$\boxed{\begin{aligned} &T(u,v) = (x,y) \\ &x = g(u,v) = x(u,v) \\ &y = h(u,v) = y(u,v) \end{aligned}}$$

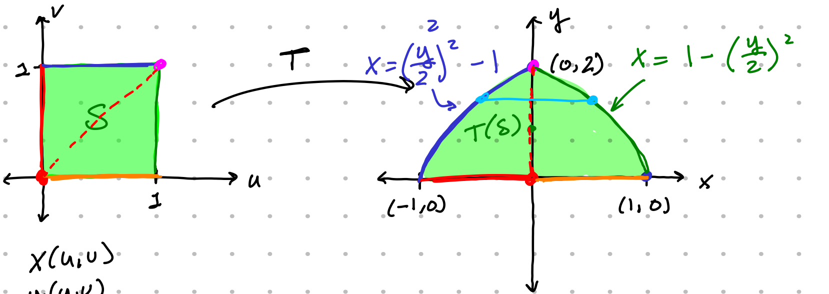
- g and h have continuous first-order partial derivatives
- T also has an inverse if it's 1-1. (+ onto)

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax+by,\ cx+dy)$$




Example: A transformation is defined by $x = u^2 - v^2$ $\quad y = 2uv$. Find the image of the square $S = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$.




$(0,0) \mapsto (0,0)$
$(0,a) \mapsto (-a^2, 0)$
$(a,0) \mapsto (a^2, 0)$ $\quad 0 \le a \le 1$
$(1,1) \mapsto (0,2)$

$x(u,v)$
$y(u,v)$

$(a,1) \mapsto (a^2 - 1,\ 2a)$

$x = a^2 - 1$ $\qquad y = 2a$

$\frac{y}{2} = a$

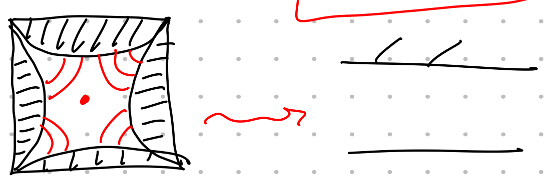
$$\boxed{\Rightarrow x = \left(\frac{y}{2}\right)^2 - 1}$$

$(1,a) \mapsto (1 - a^2,\ 2a)$

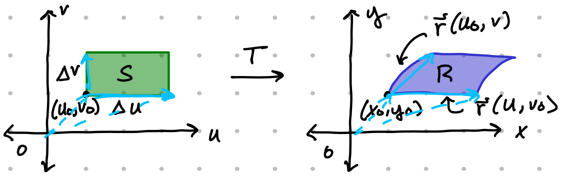
$x = 1 - a^2$ $\qquad y = 2a$

$\frac{y}{2} = a$

$$\boxed{\Rightarrow x = 1 - \left(\frac{y}{2}\right)^2}$$



How does changing variables affect the area? $\quad dxdy = dA$

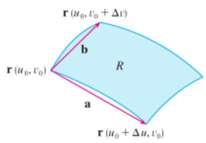


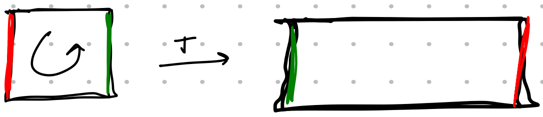

$\vec{r}(u,v) = g(u,v)\,\hat{i} + h(u,v)\,\hat{j}$

$=$ position vector of $T(u,v)$

The region $R = T(S)$ is approximately $\vec{a} \times \vec{b}$ where

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0) \quad \text{and} \quad \vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)$$




By definition of partial derivative:

$$\vec{r}_u = \lim_{\Delta u \to 0} \frac{\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)}{\Delta u}$$

$$\Rightarrow \Delta u\, \vec{r}_u \approx \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)$$

Similarly,

$$\Delta v\, \vec{r}_v \approx \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)$$

Thus, $|(\Delta u\, \vec{r}_u) \times (\Delta v\, \vec{r}_v)| = |\vec{r}_u \times \vec{r}_v|\, \Delta u\, \Delta v$

Jacobian determinant.

and $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u & \partial y/\partial u & 0 \\ \partial x/\partial v & \partial y/\partial v & 0 \end{vmatrix} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} \hat{k} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \hat{k}$

determinant

$\therefore\ dA \approx dxdy \approx |J|\, du dv$

**Definition.** The **Jacobian** of the transformation $T$ given by $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

## Change of Variables

Suppose that $T$ is

- $C^1$-transformation
- Jacobian of $T$ is non-zero
- $T$ maps $\underset{(u,v)}{S} \longmapsto \underset{(x,y)}{R}$
- $T$ is 1-1

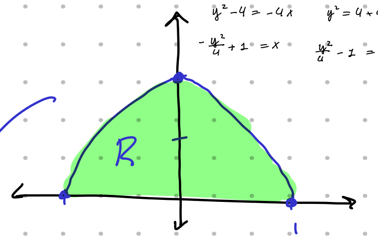
$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$T(u,v) = (x,y)$

**Example:** $x = u^2 - v^2$, $y = 2uv$. Evaluate $\boxed{\iint_R y\, dA}$ where $R$ is the region bounded by the parabolas $y^2 = 4 - 4x$ & $y^2 = 4 + 4x$, $y \geq 0$

$y^2 - 4 = -4x$ $\quad y^2 = 4+4x$

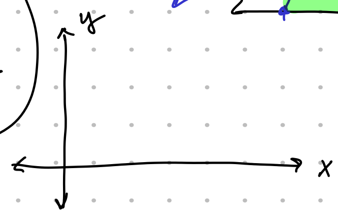
$-\frac{y^2}{4} + 1 = x$ $\quad \frac{y^2}{4} - 1 = x$




$$\iint_R y\, dA = \iint_S (2uv) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

$$\boxed{\int_0^1 \int_0^1 (2uv)(4u^2+4v^2)\, du\, dv}$$




$x = u^2 - v^2$

$y = 2uv$

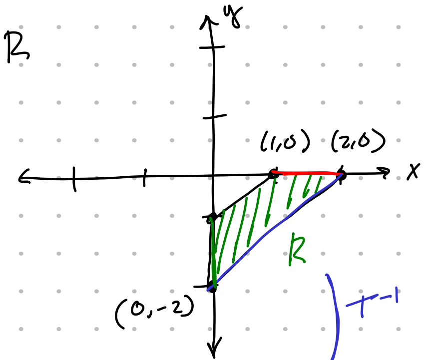
$\frac{y}{2u} = v$

$x = u^2 - \left(\frac{y}{2u}\right)^2$

write $u$ & $v$ as functions of $x$ & $y$

**Example:** Evaluate $\iint_R e^{(x+y)/(x-y)}\, dA$ $dx\,dy$ where $R$ is the trapezoidal region with vertices $(1,0)$ $(2,0)$ $(0,-2)$ $(0,-1)$

$e^{u/v}$ $\quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x+y, x-y)$




$T^{-1}$

$\boxed{u(x,y) = x+y}$ $\quad u = x+y$

$y(u,v) = \frac{1}{2}(u-v)$

$\boxed{v(x,y) = x-y}$ $\quad v = x-y$

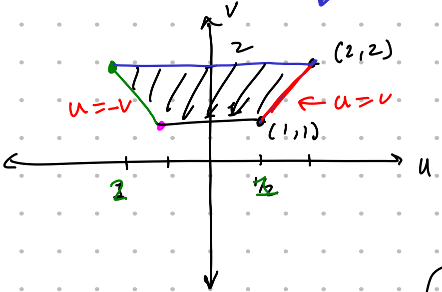
$x(u,v) = \frac{1}{2}(u+v)$

$u - y = x$

$v + y = x$

$u - v = 2y$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left|-\frac{1}{4} - \frac{1}{4}\right| = \left|-\frac{2}{4}\right| = \left|-\frac{1}{2}\right|$$




$(1,0) \longmapsto (1,1)$

$(2,0) \longmapsto (2,2)$

$(0,-2) \longmapsto (-2,2)$

$(0,-1) \longmapsto (-1,1)$

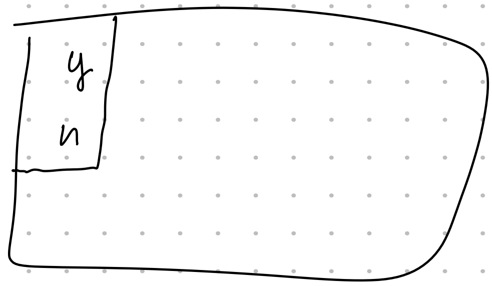
$$\boxed{\frac{1}{2} \int_1^2 \int_{-v}^{v} e^{u/v}\, du\, dv}$$

18) $\iint_R (x^2 - xy + y^2)\,dA$ where $R$ is the region bounded by the ellipse

$$x^2 - xy + y^2 = 2$$

$$x = \sqrt{2}\,u - \sqrt{2/3}\,v$$

$$y = \sqrt{2}\,u + \sqrt{2/3}\,v$$




## Examples

25) $\iint_R \cos\left(\frac{x-y}{x+y}\right) dA$ $\quad$ $R$ is trapezoid w/ $(1,0)$ $(2,0)$ $(0,2)$ $(0,1)$

26) $\iint_R \sin(9x^2 + 4y^2)\,dA$ $\quad$ $R$ is in the first quadrant bounded by $9x^2 + 4y^2 = 1$

27) $\iint_R e^{x+y}\,dA$ $\quad$ $R$ is given by $|x| + |y| \le 1$

$$\int_0^4 \int_0^{\sqrt{16-y}} f(x,y)\,dx\,dy$$

Problem 18 – 23
not on MT2

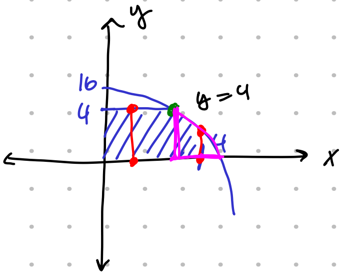
$x = \sqrt{16-y}$

$x^2 = 16 - y$

$y = 16 - x^2$

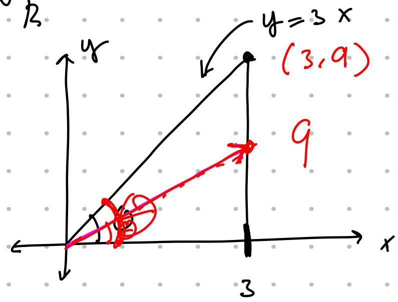
$4 - 16 = -x^2$

$-12 = -x^2$




$$\int_0^{\sqrt{12}} \int_0^4 f(x,y)\,dy\,dx + \int_{\sqrt{12}}^4 \int_0^{16-x^2} f(x,y)\,dy\,dx$$

$$\iint_R (x^2 + y^2)\,dA$$




arctan$\left(\frac{9}{3}\right)$

$$\int_0^{\arctan(9/3)} \int_0^{3/\cos(\theta)} r^2 \cdot r\,dr\,d\theta$$

$r(\theta) =$

$\cos(\theta) = \frac{3}{r}$

$r = \frac{3}{\cos(\theta)}$