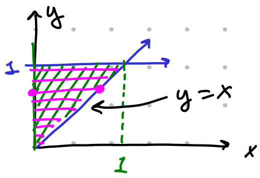
Monday, June 14, 2021

MTH 164 Lecture Notes

- Midterm 2: Monday June 21
- Times: Same as MT1.
  9:00am & 8:00pm

Example: Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$



$y = x \quad x = 0$

$y = 1 \quad x = 1$

$\iint_D \sin(y^2)\, dA$

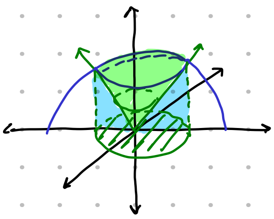
$u = y^2 \quad du = 2y\, dy \quad \frac{1}{2}du = y\, dy$

$$\int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 \sin(y^2) \cdot y\, dy = \frac{1}{2} \int_0^1 \sin(u)\, du$$

$$= \frac{1}{2}\left(-\cos(1) + \cos(0)\right)$$

$z^2 = x^2 + y^2$

Example: Find the volume above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$



$$\iint_D \sqrt{1-x^2-y^2}\, dA - \iint_D \sqrt{x^2+y^2}\, dA$$

$z = \sqrt{1-x^2-y^2}$

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le \tfrac{1}{2}\} = \{(r,\theta) : 0 \le r \le \tfrac{1}{\sqrt{2}},\ 0 \le \theta \le 2\pi\}$$

$$2(x^2+y^2) = 1 \iff x^2+y^2 = \frac{1}{2}$$

$$\boxed{\int_0^{2\pi} \int_0^{1/\sqrt{2}} \sqrt{1-r^2}\, r\, dr\, d\theta - \int_0^{2\pi} \int_0^{1/\sqrt{2}} \sqrt{r^2}\, r\, dr\, d\theta}$$

$x^2+y^2+x^2+y^2 = 1$

Section 15.6: Triple Integrals

Fubini's Theorem: If f is continuous on the rectangle $B = [a,b] \times [c,d] \times [r,s]$ then

$$\iiint_B f(x,y,z)\, dV = \left(\int_a^b \left(\int_c^d \left(\int_r^s f(x,y,z)\, dx\right) dy\right) dz\right)$$

$r \,\&\, s \longleftrightarrow$ functions of y & z

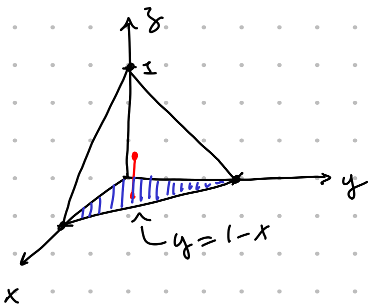
$d \,\&\, c \longleftrightarrow$ functions of z

$a \,\&\, b \longleftrightarrow$ constants.

Examples

- Evaluate $\iiint_E z\, dV$ where E is the solid tetrahedron bounded by

  $x=0,\ y=0,\ z=0$, and $x+y+z=1$



$$\int_0^1 \int_0^x \int_0^{1-x-y} z\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{1}{2} z^2 \Big|_0^{1-x-y}\right) dy\, dx$$

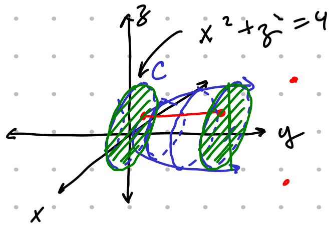
$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2\, dy\, dx$$

$A = 1-x$

$$= \int_0^1 \frac{-1}{2} \cdot \frac{1}{3}(1-x-y)^3 \Big|_0^{1-x}\, dx$$

$z = \pm\sqrt{y - x^2}$

• Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$ where E is bounded by the paraboloid $y = x^2+z^2$ & $y = 4$



$x = \sqrt{4-z^2}$

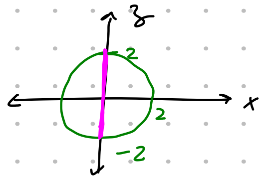
$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} \overbrace{\sqrt{r^2}\; r}^{r^2}\, dy\, dr\, d\theta$$

$C = \{(x,z) : x^2+z^2 \le 4\} = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ z &= r\sin\theta \end{aligned}}$$

$x^2+z^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$$\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dx\, dz$$

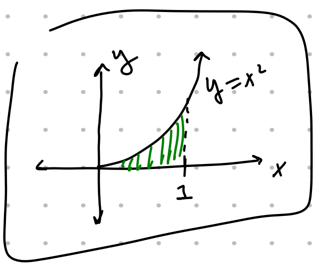


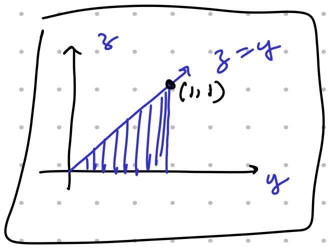
$x = \sqrt{4-z^2}$

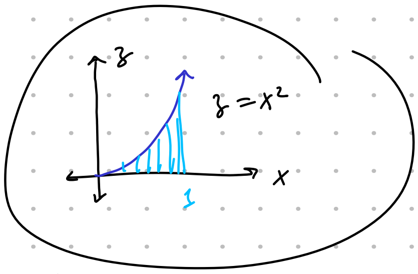
$$= \int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \sqrt{x^2+z^2}\; y \Big|_{x^2+z^2}^{4} dx\, dz = \int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \left(4\sqrt{x^2+z^2} - (x^2+z^2)\sqrt{x^2+z^2}\right) dx\, dz$$

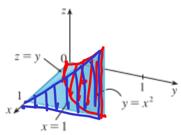
$z = 0 \qquad z = y$

• Write $\int_0^1\int_0^{x^2}\int_0^{y} f(x,y,z)\, dz\, dy\, dx$ so that dx is first, then dz, then dy





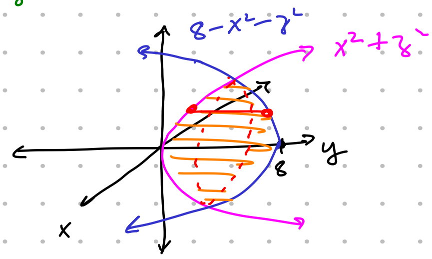




$$\int_0^1\int_0^{y}\int_{\sqrt{y}}^{1} f(x,y,z)\, dx\, dz\, dy$$

• Find the volume bounded by $y_1 = x^2+z^2$ and $y_2 = 8 - x^2 - z^2$

$$\boxed{\int_0^{2\pi}\int_0^{2}\int_{r^2}^{8-r^2} r\, dy\, dr\, d\theta}$$



$x = r\cos\theta$
$z = r\sin\theta$

$y = r^2$
$y = 8 - r^2$

$r^2 = 8 - r^2$

$r^2 = \frac{8}{2} = 4$

Break 10:20

Section 15.7: Integrals in Cylindrical Coordinates

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

(This is really just polar coordinates in 2D.)

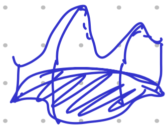
$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\; r\,dz\,dr\,d\theta$$

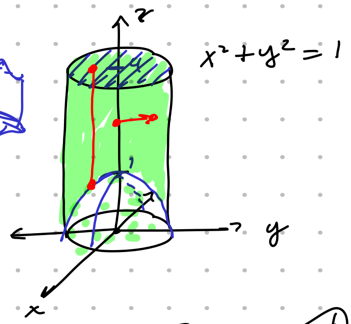
Example: A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

density $(x^2+y^2) = k\cdot(x^2+y^2)$

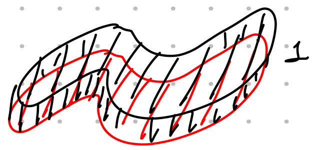
$x = r\cos\theta$
$y = r\sin\theta \quad \Rightarrow \quad d(x,y,z) = k\cdot r^2$
$z = z$

$f(x,y)$





$$\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 \overset{1}{k\cdot r^2}\cdot \boxed{r\,dz\,dr\,d\theta}\; dV \qquad \iint dA$$



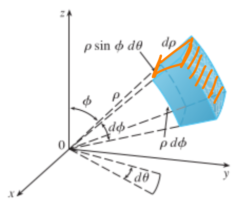
$$= k2\pi \int_0^1\int_{1-r^2}^4 r^3\,dz\,dr = k2\pi\int_0^1 r^3\cdot z\Big|_{1-r^2}^4 dr$$

$$= k2\pi\int_0^1 \left(r^3\cdot 4 - r^3(1-r^2)\right) dr$$

$$= k\cdot 2\pi\left(\underbrace{r^4 - \tfrac{1}{4}r^4}_{\frac{3}{4}r^4} + \tfrac{1}{6}r^6\right)\Big|_0^1 = \boxed{k\cdot 2\pi\left(\tfrac{3}{4}+\tfrac{1}{6}\right)}$$

($-r^3+r^5$)

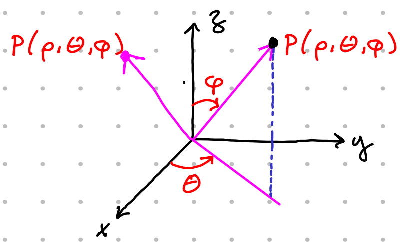
Example: Evaluate $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$ by changing to cylindrical coordinates.



Section 15.8: Triple integrals in spherical coordinates

$x = \rho\sin\varphi\cos\theta$	$\rho \ge 0$
$y = \rho\sin\varphi\sin\theta$	$0 \le \theta \le 2\pi$
$z = \rho\cos\varphi$	$0 \le \varphi \le \pi$

Be careful!



$$\boxed{\begin{aligned} dV &= dx\,dy\,dz \\ dV &= \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi \end{aligned}}$$

$$x^2+y^2+z^2 = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\varphi$$

$$= \rho^2\sin^2\varphi\left(\cos^2\theta+\sin^2\theta\right) + \rho^2\cos^2\varphi = \rho^2\left(\sin^2\varphi+\cos^2\varphi\right) = \rho^2$$

Example: Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$ where B is the unit sphere.

$$\int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\rho^3}\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$u=\rho^3 \quad du = 3\rho^2 d\rho$

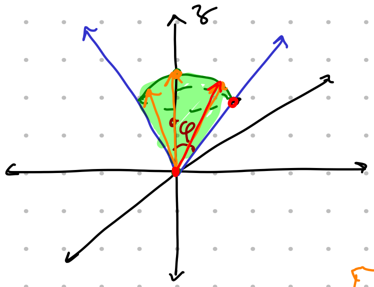
$\frac{1}{3}du = \rho^2 d\rho$

$$\int_0^{\pi}\int_0^{2\pi}\left(\int_0^1 e^u\,\frac{1}{3}du\right)\sin\varphi\,d\theta\,d\varphi$$

$$=\int_0^{\pi}\int_0^{2\pi}\left(\frac{1}{3}e-\frac{1}{3}\right)\sin\varphi\,d\theta\,d\varphi = \left(\frac{2\pi e}{3}-\frac{2\pi}{3}\right)\int_0^{\pi}\sin\varphi\,d\varphi$$

$$=\left(\frac{2\pi e}{3}-\frac{2\pi}{3}\right)(-\cos\varphi)\Big|_0^{\pi} = \boxed{2\cdot\left(\frac{2\pi e}{3}-\frac{2\pi}{3}\right)}$$

Example: Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.



$x=\rho\sin\varphi\cos\theta$

$y=\rho\sin\varphi\sin\theta$

$z=\rho\cos\varphi$

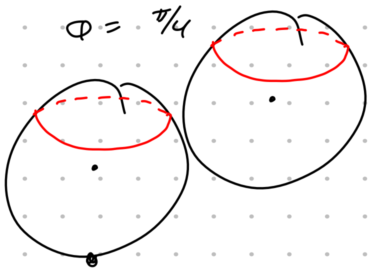
cone: $\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta}$

$=\sqrt{\rho^2\sin^2\varphi} = \rho\sin\varphi \qquad \boxed{\cos\varphi=\sin\varphi}$

$\rho^2=\rho\cos\varphi \iff \boxed{\rho=\cos\varphi}$

$$\boxed{\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi}$$

$\varphi = \pi/4$



$$2\pi\int_0^{\pi/4}\frac{1}{3}(\cos\varphi)^3\sin\varphi\,d\varphi$$

$$\int_0^{2/\sqrt{2}}\int_y^{\sqrt{4-y^2}} \underbrace{e^{3x^2+3y^2}}\,dx\,dy$$

$h(r,\theta) = e^{3r^2}$