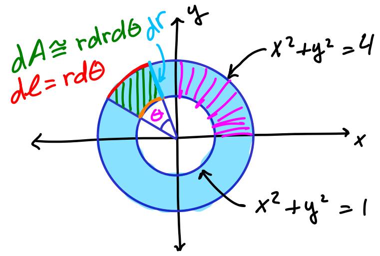
Thursday, June 10, 2021
MTH164 Lecture Notes

a) • min/max (Lagrange)
b) • Double integrals
c) • other

Section 15.3: Double Integrals in Polar Coordinates



$x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

Polar "Rectangle":

$R = \{(r,\theta): a \le r \le b,\ \alpha \le \theta \le \beta\}$

$$\begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$$

Change to Polar Coordinates:

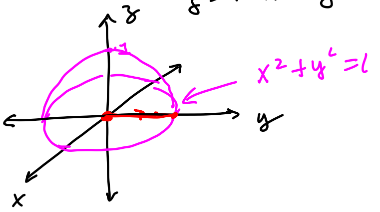
If f is continuous on a polar region R given by $0 \le a \le r$, $\alpha \le \theta \le \beta$ where $0 \le \beta - \alpha \le 2\pi$.

Then:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

↖ det(Jacobian)

Example: Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z = 1-x^2-y^2$.



$f(x,y) = 1-x^2-y^2$

$R = \{(x,y): x^2+y^2 \le 1\} = \{(r,\theta): 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$x = r\cos\theta \quad y = r\sin\theta$

$$\iint_R (1-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\cdot r\,dr\,d\theta$$

$$= 2\pi \cdot \int_0^1 (r-r^3)\,dr = \cancel{2}\pi \cdot \frac{1}{\cancel{2}} r^2\Big|_0^1 - \cancel{2}\pi \cdot \frac{1}{\cancel{4}_{2}} r^4\Big|_0^1 = \boxed{\pi - \frac{\pi}{2}} = \frac{\pi}{2}$$

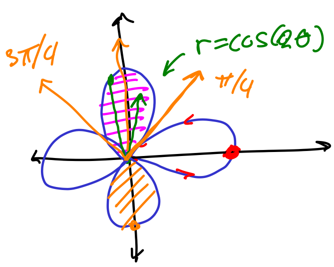
If f is continuous on $D = \{(r,\theta): \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$

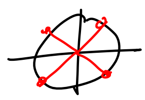
then
$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

(3θ)

$r = \cos(n\theta)$

Example: Use a double integral to find the area enclosed by one loop of the rose $r = \cos(2\theta)$.





$\pi/2$

$\cos(2\theta) = 0 \implies \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

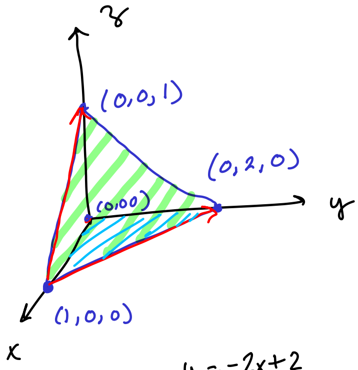
$0 \le \theta \le 2\pi$

$$\int_{\pi/4}^{3\pi/4}\int_0^{\cos(2\theta)} r\,dr\,d\theta = \boxed{\int_{\pi/4}^{3\pi/4} \frac{1}{2}\cos^2(2\theta)\,d\theta}$$

double-angle

$R = \{(r,\theta):\ 0 \le r \le \cos(2\theta)\ \ \&\ \ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$

Review of double integrals over Cartesian region.



Find the volume using a double integral

$$\iint_R f(x,y)\,dA$$

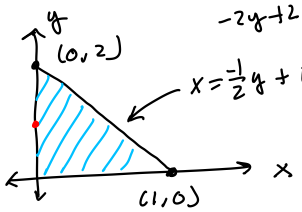
$$\int_0^1 \int_0^{-2x+2} (1-x-y/2)\; dy\,dx \qquad dV$$

$$x + \frac{y}{2} + z = 1$$

$$\boxed{z = 1 - x - \frac{y}{2}}$$

$y = -2x+2$

$-2y+2$

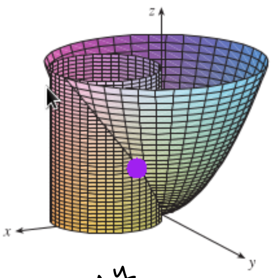


$$\boxed{\int_0^2 \int_0^{\frac{2-y}{2}} (1 - x - y/2)\, \underline{dx}\, dy}$$

$$\int_0^2 \left(\left(x - \frac{1}{2}x^2 - \frac{xy}{2}\right) \Big|_0^{\frac{2-y}{2}} \right) dy$$

$$\int_0^2 \left(\frac{2-y}{2} - \frac{(2-y)^2}{2\cdot 4} - \frac{y(2-y)}{4} \right) dy$$

Example: Find the volume of the solid under $z = x^2+y^2$ & above the xy-plane & inside the cylinder $\boxed{x^2+y^2 = 2x}$



$$\iint_R f(x,y)\,dA$$

$$x^2 - 2x + y^2 = 0$$

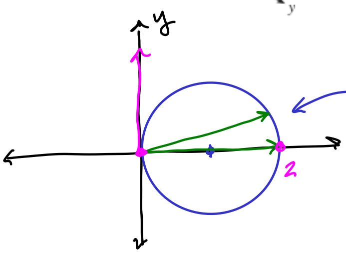
$$\boxed{(x-1)^2 + y^2 = 1}$$

$$f(x,y) = x^2+y^2$$

$$R = \{ (r,\theta) : \; 0 \le r \le 2\cos\theta \;\&\; 0 \le \theta \le 2\pi \}$$

Break.

10:10



$r(\theta) = 2\cos\theta$

$\theta(r) = \cos^{-1}\left(\frac{r}{2}\right)$

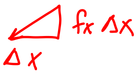
$$\boxed{r^2 = 2r\cos(\theta)}$$

$$\boxed{r = 2\cos\theta}$$

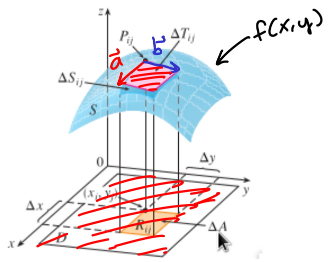
$$V = \int_0^{\pi} \int_0^{\underline{2\cos\theta}} r^2 \cdot r\,dr\,d\theta$$

$$= \boxed{\int_0^{\pi} \frac{1}{4}(2\cos\theta)^4 \; d\theta}$$

Section 15.5 : Surface Area



$$\frac{f_x \cancel{\Delta x}}{\cancel{\Delta x}}$$



Recall that $f_x(x,y)$ and $f_y(x,y)$ are slopes.

Thus

$$\vec{a} = \Delta x\,\vec{i} + f_x(x_i, y_i)\,\Delta x\,\vec{k}$$

$$\vec{b} = \Delta y\,\vec{j} + f_y(x_i, y_i)\,\Delta y\,\vec{k}$$

$$\Rightarrow \quad \Delta T_{ij} = |\vec{a} \times \vec{b}|$$

$$\boxed{\begin{array}{l} \vec{a} = \langle \Delta x,\ f_x \Delta x \rangle \\ \vec{b} = \langle \Delta y,\ f_y \Delta y \rangle \end{array}}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x(x_i,y_i) \\ 0 & \Delta y & f_y(x_i,y_i) \end{vmatrix} = \langle -f_x(x_i,y_i)\,\Delta x \Delta y,\ -f_y(x_i,y_i)\,\Delta x \Delta y,\ \Delta x \Delta y \rangle$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{(f_x(x_i,y_i))^2 + (f_y(x_i,y_i))^2 + 1}\ \Delta A$$

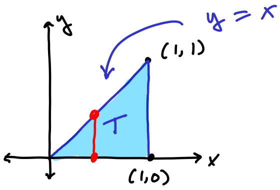
$$A(S) = \lim_{(\Delta x, \Delta y) \to (0,0)} \sum_{i,j} \sqrt{(f_x(x_i,y_i))^2 + (f_y(x_i,y_i))^2 + 1}\ \Delta A$$

Surface Area

$$A(S) = \iint_D \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\ dA$$

$$= \iint_D \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\ dA$$

Example : Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T:



$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2$$

$$\sqrt{1 + (4x^2) + 4} = \sqrt{5 + 4x^2}$$

$u = 5 + 4x^2$

$du = 8x\,dx$

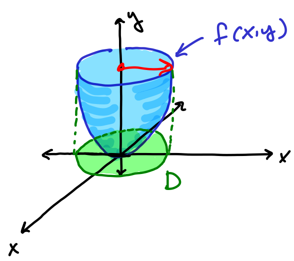
$\frac{1}{8}du = x\,dx$

$$\int_0^1 \int_0^x \sqrt{5 + 4x^2}\ dy\,dx = \int_0^1 x\sqrt{5 + 4x^2}\ dx$$

$$\boxed{\frac{1}{8}\int_5^9 \sqrt{u}\ du}$$

Example : Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under $z = 9$

$9 = x^2 + y^2$



$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

$$D = \{(r,\theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$$

$$\iint_D \sqrt{4x^2 + 4y^2 + 1}\ dA$$

$u = 4r^2 + 1$

$du = 8r\,dr$

$$\int_0^{2\pi}\int_0^3 \sqrt{4r^2 + 1}\ r\,dr\,d\theta = 2\pi \int_0^3 \sqrt{4r^2 + 1}\ r\,dr$$

$$\sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1} = \sqrt{4r^2 + 1}$$

Min/Max 14.6

$x^2+y^2=9$

1. Crit. Points.

2. Boundary.

21) Find the extreme values of f on the region

$f(x,y) = x^2+y^2+4x-4y$ $\qquad x^2+y^2 \le 9$

$f_x(x,y) = 2x+4 = 0 \Rightarrow 2x=-4 \Rightarrow x=-2$

$f_y(x,y) = 2y-4 = 0 \Rightarrow 2y=4 \Rightarrow y=2$

$\Rightarrow$ $\boxed{P_1(-2,2) \quad f(-2,2) = 4+4-8-8 = -8}$ min.

$\nabla f = \langle 2x+4, 2y-4\rangle = \lambda \nabla g = \lambda\langle 2x, 2y\rangle$

$2x+4 = \lambda 2x$

$2y-4 = \lambda 2y$ $\Rightarrow$ when $x,y \ne 0,0$

$x^2+y^2=9$

$\frac{2x+4}{x} = \frac{2y-4}{y}$

$\Rightarrow 2xy+4y = 2xy-4x$

$x^2+y^2 = 2x^2 = 9$

$\Rightarrow 4y=-4x \Rightarrow y=-x$

$\Rightarrow x = \pm\sqrt{9/2}$

$\left(3/\sqrt{2}, -3/\sqrt{2}\right)$ $\left(-3/\sqrt{2}, 3/\sqrt{2}\right)$

$f\left(3/\sqrt{2}, -3/\sqrt{2}\right) =$

$\boxed{\underbrace{\frac{9}{2}+\frac{9}{2}}_{9} + \frac{4\cdot 3}{\sqrt{2}} + \frac{4\cdot 3}{\sqrt{2}}} > 0$ max

$f\left(-3/\sqrt{2}, 3/\sqrt{2}\right) =$

$\frac{9}{2}+\frac{9}{2} - \frac{2(\overset{12}{4\cdot 3})}{\sqrt{2}}$

When the 2nd derivative test fails.

$f(x,y) = x^4+y^4-x^3$

$4x-3=0$

$f_x = 4x^3-3x^2 = 0 \Rightarrow$ either $x=0$ or $x=3/4$

$f_y = 4y^3 \Rightarrow y=0$

Crit Points

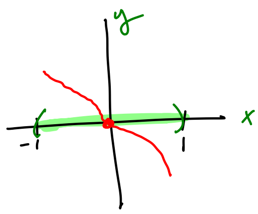
$(0,0)$ & $(3/4, 0)$

$(0,0) \rightsquigarrow$ saddle point.

when $x=0$, $f(x,y) = f(y) = y^4 \rightsquigarrow$ concave up.

when $y=0$ $\quad f(x,y) = f(x) = x^4-x^3$

"Zoom in" to interval $(-1,1)$ around $(0,0)$.



- Let $\frac{1}{A}$, $A>1$, be an arbitrary point in $(0,1)$

  then $f(1/A) = \frac{1}{A^4} - \frac{1}{A^3} < 0 \quad \forall$ such A

- Let $\frac{-1}{A}$, $A>1$, be an arbitrary point in $(0,1)$

  then $f(-1/A) = \frac{1}{A^4} + \frac{1}{A^3} > 0 \quad \forall$ such A

$\Rightarrow$ inflection point.

$\therefore$ $(0,0)$ is a saddle point for $f(x,y)$.

What about $(3/4, 0)$? $f(x,y) = x^4 + {}^4 - x^3$

We know from inspection that $f(x,y)$ must have a min.
Thus, since $(0,0)$ is not a min, $(3/4,0)$ is.

$f_{xx} = 12x^2 - 6x$ $\quad f_{xy} = 0$

$f_{yy} = 12y$

$\Rightarrow D(3/4, 0) = \left(12 \cdot (3/4)^2 - 6(3/4)\right)(0) - 0 = 0$

$f_x = 4x^3 - 3x^2$

$f_y = 4y^3$ $\quad f_{xy}$

Note that $\boxed{\lim_{(x,y) \to (\pm\infty, \pm\infty)} f(x,y) = +\infty.}$

& furthermore, $f(x,y)$ is cont. &

$\Rightarrow f(x,y)$ must have a min

$$\boxed{f(x,y) = \frac{1}{1 + x^2y^2}}$$

$f(x,y) = \cos(xy)$

$f_x = -y\sin(xy)$

$f_{xx} = -y^2\cos(xy)$

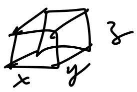
$f_y = -x\sin(xy)$

$f_{yy} = -x^2\cos(xy)$

$f_{xy} = y^3\sin(xy) - 2y\cos(xy)$

$\Rightarrow$ 2nd d.t. fails.

$\boxed{V = 42 \text{ cm}^3}$ what are lengths of edges s.t. SA is at a min



$V(x,y,z) = x \cdot y \cdot z = 42$

$SA(x,y,z) = 2xy + 2yz + 2xz$

Minimize SA: $z = \frac{42}{xy}$ (what if $x,y = 0$?)

$f(x,y) = 2xy + \frac{2\cancel{y} \cdot 42}{x\cancel{y}} + \frac{2\cancel{x} \cdot 42}{\cancel{x}y}$

$= 2xy + \frac{84}{x} + \frac{84}{y}$

$\frac{\partial f}{\partial x} = 2y - \frac{84}{x^2} = 0$ $\quad \frac{\partial f}{\partial y} = 2x - \frac{84}{y^2} = 0$

$2y = \frac{84}{x^2}$ $\quad 2x = \frac{84}{y^2}$ $\quad 2yx^2 = 84$ $\quad 2xy^2 = 84$ $\Rightarrow \cancel{2}y^2\cancel{x} = \cancel{2}x^2\cancel{y} \Rightarrow \boxed{y = x}$

$2x^3 = 84$

$x^3 = 42 = y^3$

$z = \frac{42}{(42)^{2/3}}$

4) $xy + \underline{12x} + z^2 = 144$ Shortest distance to origin.

Minimize $d(x,y,z) = x^2 + y^2 + z^2$

$$d(x,y) = x^2 + y^2 + 144 - xy - 12x$$

$$d_x = 2x - y - 12 = 0$$

$$d_y = 2y - x = 0 \implies x = 2y$$

$$4y - y - 12 = 3y - 12 = 0 \implies y = 12/3 = 4 \implies x = 8$$

$$32 + 12 \cdot 8 + z^2 = 144$$

$$z^2 = 144 - 32 - 12 \cdot 8$$