Wednesday, June 9, 2021
MTH 164 Lecture Notes

Some extra practice from yesterday.

1. $f(x,y) = x^2+y^2+xy$

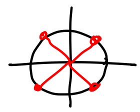
3. $f(x,y) = e^{-xy}$, $x^2+4y^2 \le 1$

Find the local max & mins

$f_x(x,y) = 2x + y = 0$
$f_y(x,y) = 2y + x = 0$ $\Rightarrow (0,0)$

$f_{xx}(x,y) = 2$
$f_{yy}(x,y) = 2$ $\Rightarrow$ $D = 4-1 = 3 > 0$, $f_{xx} = 2 > 0$ $\Rightarrow$ $(0,0)$ is a minimum
$f_{xy} = 1$

Find the extreme values of $f(x,y)$ subject to the constraint $x^2+y^2=1$



parameterize: in $\mathbb{R}^2$, the constraint is $\vec{r}(t) = \langle \cos t, \sin(t) \rangle$, $\boxed{0 \le t < 2\pi}$
($\cos t = x$, $\sin(t) = y$)

$\Rightarrow f(x(t), y(t)) = \cos^2 t + \sin^2 t + \cos(t)\sin(t)$
$= 1 + \cos(t)\sin(t)$

$\frac{d}{dt} f(t) = \cos^2(t) - \sin^2(t) = 0 \Rightarrow \cos^2(t) = \sin^2(t)$
$\cos(t) = \pm\sin(t)$

$\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$f(x,y) = x^2+y^2+xy$

$f(\sqrt{2}/2, \sqrt{2}/2) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} = f(-\sqrt{2}/2, -\sqrt{2}/2)$

$f(\sqrt{2}/2, -\sqrt{2}/2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2} = f(-\sqrt{2}/2, \sqrt{2}/2)$

Using Lagrange multipliers.

$f(x,y) = x^2+y^2+xy$, $g(x,y) = x^2+y^2-1$

$\nabla f = \lambda \nabla g$
$2x + y = \lambda \cdot 2x$
$2y + x = \lambda \cdot 2y$
$x^2+y^2=1$

$\boxed{\text{when } x+y \neq 0:}$

$\frac{2x+y}{x} = \frac{2y+x}{y} \Rightarrow \cancel{2xy} + y^2 = \cancel{2xy} + x^2$

$\Rightarrow y^2 = x^2 \Rightarrow x = \pm y$

$\Rightarrow$ Solutions are at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$

So we get the same answer.

2. $f(x,y) = x^4 + y^4 - x^3$ Find & classify the critical points.

$f_x(x,y) = 4x^3 - 3x^2 = 0$ $4x^3 = 3x^2$ if $x \neq 0$, $4x = 3 \Rightarrow x = \frac{3}{4}$, $x = 0$

$f_y(x,y) = 4y^3 = 0$ $y = 0$

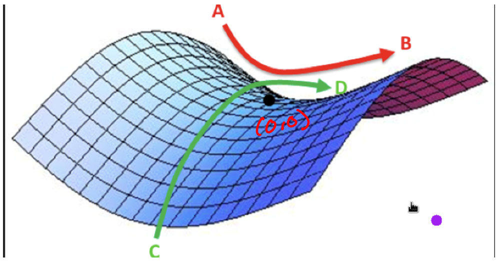
$\boxed{\text{Critical points: } (3/4, 0) \ \& \ (0,0)}$

$f_{xx}(x,y) = 12x^2 - 6x$

$f_{yy}(x,y) = 12y^2$

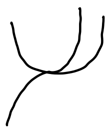
$f_{xy} = 0$

$D(0,0) = 0 \Rightarrow 2^{nd}$ derivative test is inconclusive!



Find a path that passes through (0,0) & test concavity.
If you can find two paths, $r_1(t)$ & $r_2(t)$ that pass through (0,0) but have different concavity, we know it's a saddle point.

$D < 0 \iff$ saddle.

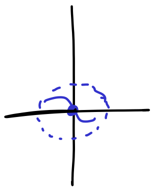


when $y = 0$, $f(x,0) = x^4 - x^3 \leftarrow (0,0)$ is an inflection point

when $x = 0$ $f(0,y) = y^4 \leftarrow (0,0)$ is a minimum.

$f(x,0)$ is negative when $x^4 < x^3$
positive when $x^4 > x^3$

$\therefore$ (0,0) is a saddle point



1. (0,0) is crit. point.
2. $f(0) = 0$
3. $f(-\varepsilon) > 0$
4. $f(\varepsilon) < 0$

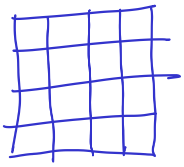
$\Rightarrow$ Inflection point.

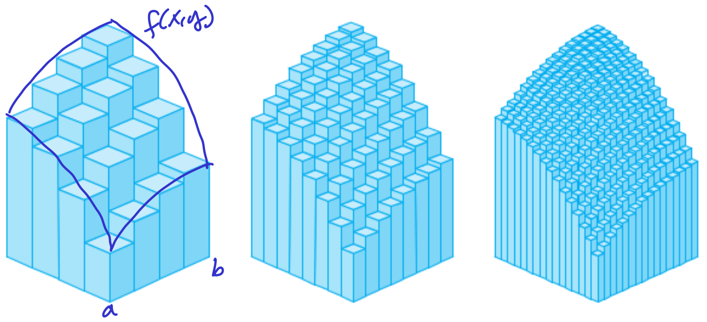
$f(\frac{1}{b}, 0) < 0$ when $\frac{1}{b} < 1$ $b > 1$

$\left(\frac{1}{b}\right)^4 - \left(\frac{1}{b}\right)^3 = \frac{1}{b^4} - \frac{1}{b^3} < 0$

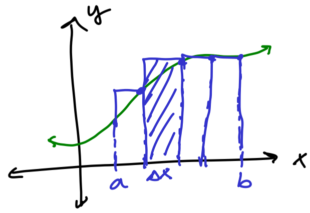
$f(-\frac{1}{b}, 0)$

Section 15.1: Double integrals over rectangular regions

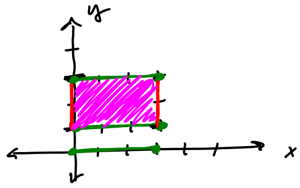




(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$



Example: $\left(\int_0^3 \left(\int_1^2 x^2 y \, dy\right) dx\right)$



$\int_1^2 \int_0^3 x^2 y \, dx \, dy$

$x^2 \int_1^2 y \, dy = x^2 \cdot \frac{1}{2} y^2 \Big|_1^2 = x^2 \cdot \frac{1}{2}(4-1) = x^2 \cdot \frac{3}{2}$ $\quad \int_0^3 x^2 \cdot \frac{3}{2} \, dx = \frac{3}{2} \cdot \frac{1}{3} x^3 \Big|_0^3 = \frac{1}{2} \cdot 3^3 = \boxed{\frac{27}{2}}$

$\int_1^2 \int_0^3 x^2 \cdot y \, dx \, dy = \int_1^2 y \, \frac{1}{3} x^3 \Big|_0^3 \, dy = \int_1^2 y \cdot 9 \, dy = \frac{9}{2} \cdot y^2 \Big|_1^2 = \frac{9}{2}(4-1) = \frac{9 \cdot 3}{2} = \boxed{\frac{27}{2}}$

10 Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

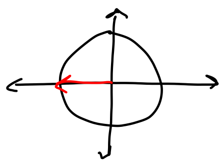
$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

$dA = dx\,dy$ or $dy\,dx$

Example: $\iint_R (x - 3y^2) \, dA$ $\quad R = \{(x,y) : 0 \leq x \leq 2 \text{ \& } 1 \leq y \leq 2\} = [0,2] \times [1,2]$

$$\boxed{\int_1^2 \int_0^2 (x - 3y^2) \, dx \, dy}$$



cartesian product.

Example: $\iint_R y \sin(xy) \, dA$ $\quad R = \underset{x}{[1,2]} \times \underset{y}{[0,\pi]}$ $\quad$ (doing dx first is easier)

$= \{(x,y) \in \mathbb{R}^2 : x \in [1,2], y \in [0,\pi]\}$

$\int_1^2 \int_0^\pi y \sin(xy) \, dy \, dx$ $\qquad \frac{\partial}{\partial x} \sin(xy)$

$\int_0^\pi \int_1^2 y \sin(xy) \, dx \, dy = \int_0^\pi \cancel{y} \left(\frac{-1}{\cancel{y}} \cos(xy) \Big|_1^2 \right) dy = \int_0^\pi (-\cos(2y) + \cos(y)) \, dy$

$= -\frac{1}{2} \sin(2y) \Big|_0^\pi + \sin(y) \Big|_0^\pi = 0$

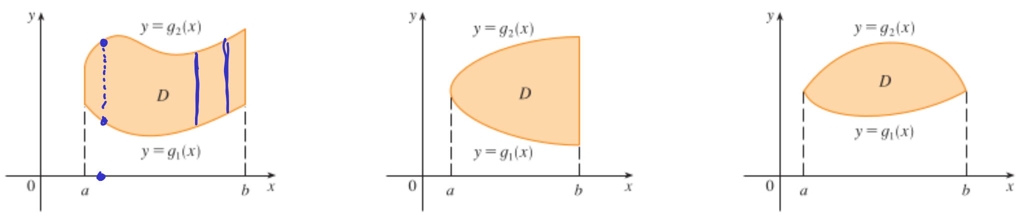
11 $\iint_R g(x) \, h(y) \, dA = \int_a^b g(x) \, dx \int_c^d h(y) \, dy \quad$ where $R = [a, b] \times [c, d]$

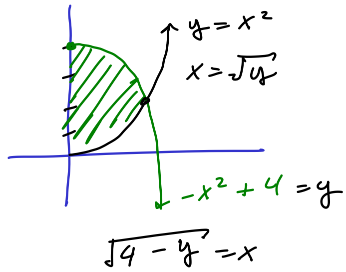
Break: Back at 10:17

Section 15.2: Double integrals over general regions

$\int_0^a \int_{x^2}^{-x^2+4} f(x,y)\, dy\, dx$

"Type I region"





"Type II regions" $\int_c^d \int_a^b \, dx\, dy$

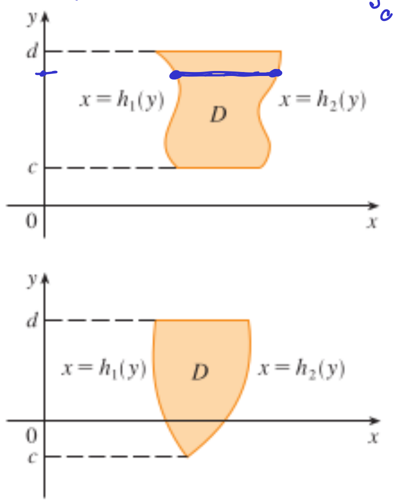


FIGURE 7
Some type II regions

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

then $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

$\int_{g_1(x)}^{g_2(x)} \int_a^b f(x,y)\, dx\, dy$

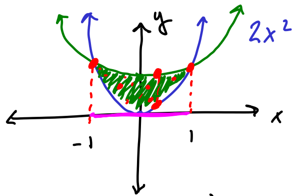
5 $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

Example: Evaluate $\iint_D (x+2y)\, dA$ where D is the region bounded by $y = 2x^2$ & $y = 1 + x^2$

$D = \{(x,y) \mid \ \leq x \leq$

$2x^2 \leq y \leq 1 + x^2$



$$\boxed{\int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx}$$

$2x^2 = 1 + x^2$

$x^2 = 1 \implies x = \pm 1$

$$\int_{-1}^{1} \left(\int_{2x^2}^{1+x^2} x\, dy + \int_{2x^2}^{1+x^2} 2y\, dy \right) dx$$

$$= \int_{-1}^{1} \left(x(y)\Big|_{2x^2}^{1+x^2} + 2\,\frac{1}{2}\cdot y^2 \Big|_{2x^2}^{1+x^2} \right) dx$$

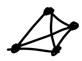
$$= \int_{-1}^{1} \Big[x\left(1 + x^2 - 2x^2\right) + \underbrace{(1+x^2)^2}_{1+x^4+2x^2} - \underbrace{(2x^2)^2}_{4x^4} \Big]\, dx$$

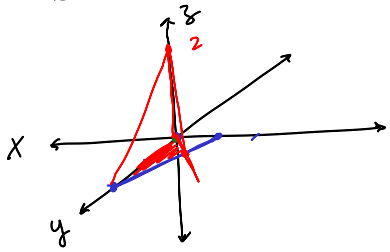
$$= \int_{-1}^{1} (x - x^3 + 1 - 3x^4 + 2x^2)\, dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

Example 4: Find the volume of the ~~tetrahedron~~ shape bounded by the planes
4

$$\begin{cases} x+2y+z=2 \\ x=2y \\ x=0 \\ z=0 \end{cases}$$

$$\iint_D f(x,y)\,dA$$





$f(x,y) = z = 2 - x - 2y$

$y = \frac{1}{2}x$

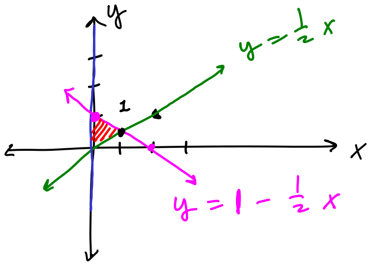
$x + 2y = 2$

$2y = 2 - x$

$y = 1 - \frac{1}{2}x$

$\frac{1}{2}x = 1 - \frac{1}{2}x$

$x = 1$

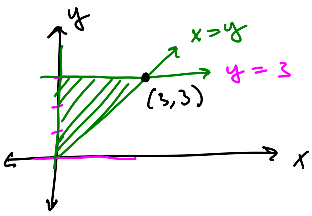


$$\int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} (2-x-2y)\,dy\,dx$$

$$\int_0^1 2y - xy - y^2 \Big|_{1/2x}^{1-1/2x} dx$$

$$= \int_0^1 2\left(1-\tfrac{1}{2}x - \tfrac{1}{2}x\right) - x\left(1-\tfrac{1}{2}x - \tfrac{1}{2}x\right) - \left(1-\tfrac{1}{2}x\right)^2 - \left(\tfrac{1}{2}x\right)^2 dx$$

a) $\iint_D e^{-y^2}\,dA \qquad D = \{(x,y) : 0 \le y \le 3,\ 0 \le x \le y\}$



$$\int_0^3 \int_0^y e^{-y^2}\,dx\,dy$$

$$= \int_0^3 e^{-y^2} \cdot \underline{y\,dy} \qquad u = -y^2 \qquad du = -2y\,dy$$

$$-\tfrac{1}{2}du = y\,dy$$

$$= \frac{1}{2}\int_{-9}^{0} e^u\,du = \frac{1}{2}\left(e^0 - e^{-9}\right)$$

23–32 Find the volume of the given solid.

23. Under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

24. Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

25. Under the surface $z = xy$ and above the triangle with vertices $(1, 1)$, $(4, 1)$, and $(1, 2)$

26. Enclosed by the paraboloid $z = x^2 + y^2 + 1$ and the planes $x = 0$, $y = 0$, $z = 0$, and $x + y = 2$

27. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

28. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

29. Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

30. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

31. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

32. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$