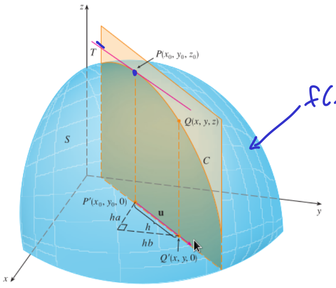
Monday June 7, 2021
MTH 164 Lecture Notes

Section 14.6: Directional Derivatives and Gradient Vectors

$f_x(x,y)$



$f(x,y)$

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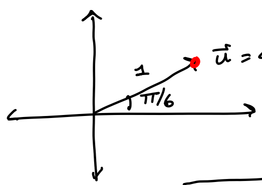
Theorem: $D_{\vec{u}}f(x,y) = \underline{f_x(x,y)}\,a + \underline{f_y(x,y)}\,b$ $\qquad \dfrac{\vec{v}}{|\vec{v}|}$

where $\vec{u} = \langle a, b\rangle$ $\boxed{\vec{u} \text{ must be a unit vector}}$

Example: Find the directional derivative $D_{\vec{u}}f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$, $\vec{u}$ is the unit vector given by $\theta = \pi/6$. What is $D_{\vec{u}}f(1,2)$?



$\vec{u} = \langle \cos(\pi/6), \sin(\pi/6)\rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

$f_x(x,y) = 3x^2 - 3y \qquad f_x(1,2) = 3 - 6 = -3$

$f_y(x,y) = -3x + 8y \qquad f_y(1,2) = -3 + 16 = 13$

$$\boxed{D_{\vec{u}}f(1,2) = -3\cdot\frac{\sqrt{3}}{2} + 13\cdot\frac{1}{2}}$$

Definition: If $f(x,y)$ is a function, then the $\underline{\text{gradient}}$ of f to be the vector function ∇f:

"nabla"

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$\Rightarrow \boxed{D_{\vec{u}}f(x,y) = \nabla f(x,y)\cdot\vec{u}}$$

Example: $f(x,y,z) = x\cdot\sin(yz)$ $\qquad \vec{v} = \langle 1,2,-1\rangle$ @ $(1,3,0)$

$\nabla f = \langle \sin(yz), zx\cos(yz), yx\cos(yz)\rangle$

$\nabla f(1,3,0) = \langle 0, 0, 3\rangle$

$|\vec{v}| = \sqrt{6}$

$\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt{6}}\langle 1,2,-1\rangle = \vec{u}$

$$\boxed{D_{\vec{u}}f(1,3,0) = \langle 0,0,3\rangle\cdot\frac{1}{\sqrt{6}}\langle 1,2,-1\rangle}$$

□ Maximizing the Directional derivative.

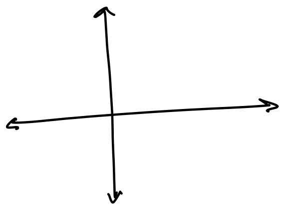
$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| \cdot |\vec{u}| \cdot \cos\theta$$

$$= |\nabla f| \cdot \cos\theta$$

When is this at a max?

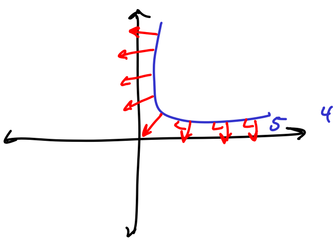
$\theta = 0$

$$\frac{\nabla f(a,b)}{|\nabla f|}$$



theorem: Suppose f is a diff. function of 2 or 3 variables. The max value of $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ & it occurs when $\vec{u}$ is in the direction of ∇f.

□ tangent planes to level surfaces.



$f(x,y) = 3$

* tangent plane to the level surface $F(x,y,z) = k$ at $P(x_0, y_0, z_0)$ is perpendicular to $\nabla F(x_0, y_0, z_0)$.

Tangent Plane of $F(x,y,z) = k$ is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$= \nabla F(x_0,y_0,z_0) \cdot (\vec{r} - \vec{r}_0)$$