Thursday June 6, 2021

MTH 164 Lecture Notes

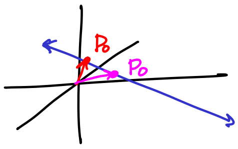
$r(t) = r_0 + t\vec{v}$

Review!

Problem 2: Find a vector equation for the line that passes through $P_0(2,4,3)$ and is parallel to $\vec{v} = \langle 1,4,-2 \rangle$

$\vec{r}(t) = \langle 2,4,3 \rangle + t\langle 1,4,-2 \rangle$

$\vec{r}(t) = \langle 2+t, 4+4t, 3-2t \rangle$



$r_0 = \langle 2,4,3 \rangle$

$\vec{v} = \langle 1,4,-2 \rangle t$

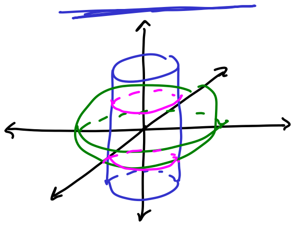
(x,y,z)

$(r_0 - r)\cdot n = 0$

Problem 1: Find a vector equation that describes the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $x+z=3$ $z = 3-x$

$\vec{r}(t) = \langle \cos(t), \sin(t), 3-\cos(t) \rangle$

Find a vector equation that describes the curve of intersection of $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$



$1 + z^2 = 4$

$z^2 = 3$

$z = \pm\sqrt{3}$

$$\boxed{\begin{aligned} \vec{r}_1(t) &= \langle \cos(t), \sin(t), +\sqrt{3} \rangle \\ \vec{r}_2(t) &= \langle \cos(t), \sin(t), -\sqrt{3} \rangle \end{aligned}}$$

$z = (x+2y-3)$

$z^2 = (x+2y-3)^2 = x^2+4y^2+4xy-12y-6x+9$

$(x+2y-3)$

$\times$

$-3x-6y+9$

$2xy+4y^2-6y$

$x^2+2xy-3x$

$x^2+y^2-x^2-4y^2-4xy+12y+6x=10$

$-3y^2-4xy+12y+6x=10$

Find a vector equation that describes the curve of intersection of $x^2+y^2-z^2=1$ with $x=z$,

$y^2=1$

$\vec{r}(t) = \langle t, \pm 1, t \rangle$

"ruled surface"

44, Section 13.1) The paraboloid $z = 4x^2+y^2$ + parabolic cylinder $y = x^2$

$z = 4x^2 + x^4$

$z = 4y + y^2$

$x = t$

$$\boxed{r(t) = \langle t, t^2, 4t^2+t^4 \rangle}$$

45) $z = x^2 - y^2 \qquad x^2 + y^2 = 1$

$\vec{r}(t) = \langle \sin(t), \cos(t), \sin^2(t) - \cos^2(t) \rangle$

0 dot & cross
1 lines & planes
2 space curves
3 derivatives & integrals of vector equations
4 arc length & curvature
5 limits & functions of several variables

Find the curvature of a circle of radius r in the xy-plane.

13.3

$\vec{r}(t) = \langle r\cos t, r\sin(t), 0 \rangle$

Theorem 10:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

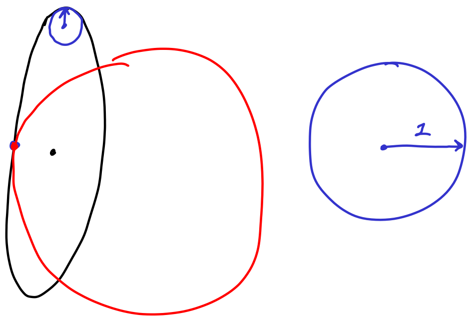
$\vec{r}'(t) = \langle -r\sin(t), r\cos t, 0 \rangle$

$\vec{r}''(t) = \langle -r\cos(t), -r\sin t, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ \langle -r\sin(t), & r\cos t, & 0\rangle \\ \langle -r\cos(t), & -r\sin t, & 0 \rangle \end{vmatrix} = i(0) - j(0) + k(r^2\sin^2 t + r^2\cos^2 t) = \langle 0, 0, r^2 \rangle$$

$|\vec{r}'(t)| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r \qquad \kappa(t) = \frac{r^2}{r^3} = \frac{1}{r}$ ✓

Given a circle of radius 1 & an ellipse of max radius 100 & min radius .5, which has a larger max curvature?



ellipse.

100, $\frac{1}{2}$

$\vec{r}(t) = \langle r\cos t, R\sin(t), 0 \rangle$

$\vec{r}'(t) = \langle -r\sin(t), R\cos t, 0 \rangle$

$\vec{r}''(t) = \langle -r\cos(t), -R\sin t, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ \langle -r\sin(t), & R\cos t, & 0\rangle \\ \langle -r\cos(t), & -R\sin t, & 0 \rangle \end{vmatrix} = k(r^2\sin^2 t + R^2\cos^2 t)$$

$R >>> r$

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{(r^2\sin^2 t + R^2\cos^2 t)}{(r^2\sin^2(t) + R^2\cos^2(t))^{3/2}} = \frac{1}{(r^2\sin^2(t) + R^2\cos^2(t))^{1/2}}$$

$t = \pi/2 \qquad r^2 \qquad \overset{\shortparallel}{0}$

$$\frac{1}{(1/2)^{1/2}} = (2)^{1/2}$$

Describe the level surfaces

$f(x,y,z) = k \implies$ surfaces

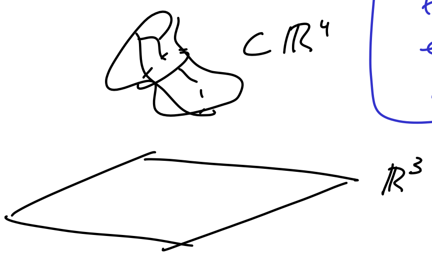
$f(x,y,z) = x^2 - y^2 - z^2$

$x^2 - y^2 - z^2 = k$

$x^2 - y^2 - z^2 = 0$

$x^2 = y^2 + z^2 \rightsquigarrow$ cone.

$-x^2 + y^2 + z^2 = k$

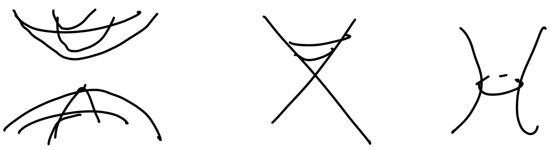


The level surfaces are a family of hyperboloids of two sheets of the form $x^2 - y^2 - z^2 = k$ except when $k=0$, in which case we have a cone.

trace : $y=0$

$x^2 = z^2$

$x = \pm z$



$$x^2 - y^2 - z^2 = k \quad \begin{cases} k > 0 & \text{Hyperboloid of two sheets} \\ k = 0 & \text{cone} \\ k < 0 & \text{hyperboloid of 1 sheet} \end{cases}$$

$$\frac{-x^2}{k} + \frac{y^2}{k} + \frac{z^2}{k} = 1$$

<u>Math Stack exchange</u> curves of intersection.

14.2

18) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

Let $x = y^4$

$$\lim_{y\to 0} \frac{y^8}{2\cdot y^8} = \frac{1}{2}$$

Let $x=0$

$$\lim_{y\to 0} 0 = 0 \qquad 0 \neq 1/2 \implies \text{DNE}$$

10) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$

Let $y=x$

$$\lim_{x\to 0} \frac{5\cancel{x^4}\cos^2 x}{2\cancel{x^4}} = \lim_{x\to 0} \frac{5}{2}\cos^2 x = \frac{5}{2}$$

Let $y=0$

$$\lim_{x\to 0} 0 = 0$$

$0 \neq 5/2 \implies$ DNE

$$\lim_{(x,y)\to(0,0)} \frac{-e^{xy}+1}{xy}$$

$x=y$ $\quad \lim_{x\to 0} \frac{-e^{x^2}+1}{x^2} = \lim_{x\to 0} \frac{-\cancel{2x}\,e^{x^2}}{\cancel{2x}} = \lim_{x\to 0} -e^{x^2} = -1$

$x=1$

$\lim_{y\to 0} \frac{-e^{y}+1}{y} = \lim_{y\to 0} \frac{-e^{y}}{1} = -1$

$y=x^2$

$\lim_{x\to 0} \frac{-e^{x^3}+1}{x^3} = \frac{-\cancel{3x^2}\,e^{x^3}}{\cancel{3x^2}} = \lim_{x\to 0} -e^{x^3} = -1$

$x=r\cos\theta \quad y=r\sin\theta$

$$\lim_{r\to 0} \frac{-e^{r^2\cos\theta\sin\theta}+1}{r^2\cos\theta\sin\theta}$$

Substitution. Let $z=xy$

$\lim_{z\to 0} \frac{-e^{z}+1}{z} = -1$

linearization

$f(x,y) = 3x^2+y^3+5y+2x+1$

linearize $f(x,y)$ at $(0,0)$ $\quad L(x,y)$

Find the tangent plane $f(x,y)$ at $(0,0,1)$

$$(z-z_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$f_x = 6x+2 \qquad f_x(x_0,y_0) = 2$

$f_y = 3y^2+5 \qquad f_y(x_0,y_0) = 5$

$(z-1) = 2x+5y$

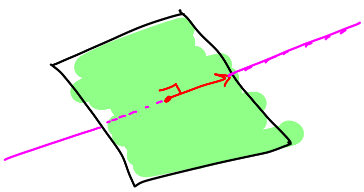
$z = 2x+5y+1$

Equations of lines & planes

$n = \langle 1,3,1\rangle$

The line through the point $\overset{r_0}{(1,0,6)}$ & perpendicular to the plane $x+3y+z=5$

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$



$\vec{r}(t) = \langle 1,0,6\rangle + t\langle 1,3,1\rangle$

$= \langle 1+t, 3t, 6+t\rangle$

$$P \quad Q \quad R$$

Plane through the points $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ $\qquad (\vec{r}-\vec{r}_0)\cdot\vec{n} = 0$

$$\begin{array}{r} \overrightarrow{PQ} = \\ \times \\ \overrightarrow{PR} = \end{array} \begin{vmatrix} i & j & k \\ \langle 1, & -1, & 0\rangle \\ \langle 1, & 0, & -1\rangle \end{vmatrix} = i(1) - j(-1) + k(1) = \langle 1, 1, 1\rangle$$

$$\langle x-0, y-1, z-1\rangle \cdot \langle 1, 1, 1\rangle = 0$$

$$x + y - 1 + z - 1 = 0$$

$$\Rightarrow \boxed{x+y+z = 2}$$

Find the equation for the line of intersection

$$x+y+z = 2$$

$$x+y = 1$$

1) point in the intersection

2) direction vector

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, 0\rangle = \vec{v}$$

$$y = z = 0 \Rightarrow x = 2$$

$$1 + z = 2 \Rightarrow z = 1$$

Point $P(0, 1, 1)$

$$\langle x, y-1, z-1\rangle \cdot \langle -1, 1, 0\rangle = 0$$