Wednesday, June 2, 2021
MTH 164 Lecture Notes

Section 14.3: Partial Derivatives

- To find the partial derivative $f_{x_i} = \frac{\partial}{\partial x_i} f(x_1, \dots, x_n) = \frac{\partial f}{\partial x_i} = \partial_i f$ of $f(x_1, \dots, x_n)$, regard x_j as constants for $j \neq i$ & differentiate with respect to x_i

- To find $f_{x_i x_j} = \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_n) = \frac{\partial^2 f}{\partial x_i \partial x_j} = \partial_{ij} f$ of $f(x_1, \dots, x_n)$, apply the process twice.

- $f(x,y) = x^3 + x^2y^3 - 2y$

Partial derivative wrt. x:
$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3$$

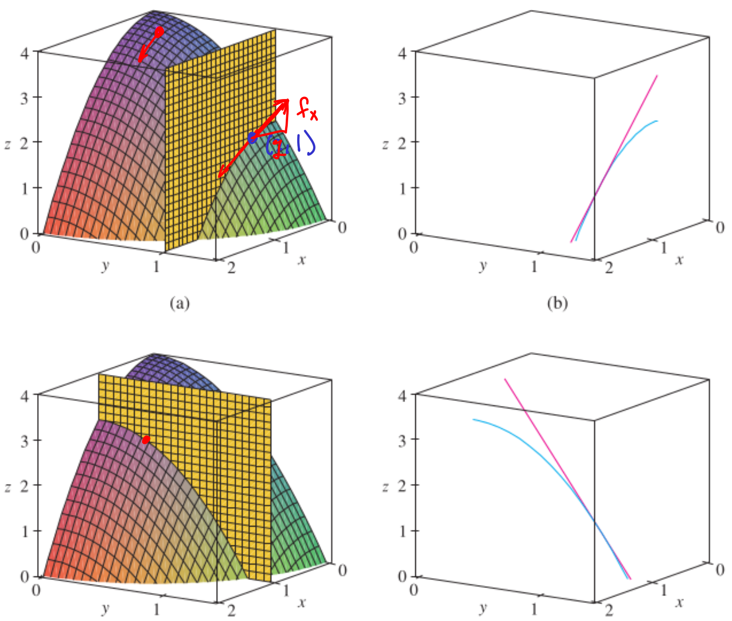
Partial derivative wrt. y:
$$\frac{\partial f}{\partial y} = 3x^2y^2 - 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

- $f(x,y) = 4 - x^2 - 2y^2$

Partial derivative wrt. x:
$$\frac{\partial f}{\partial x} = -2x \qquad \frac{\partial^2 f}{\partial x^2} = -2$$

As slopes:



at $(2,1)$, $f(x,y) = 4 - x^2 - 2y^2$
the slope is $\frac{\partial f}{\partial x}(2,1) = -4$

$\boxed{\text{Clairaut's Theorem: Suppose } f \text{ is defined on a disk } D \text{ that contains the point } (a,b). \text{ If the functions } f_{xy} \text{ \& } f_{yx} \text{ are both continuous on } D, \text{ then } f_{xy} = f_{yx}}$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, $f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$

$$f(x,y) = \ln(xy)$$

$$\frac{\partial f}{\partial x} = \frac{1}{x} \qquad \frac{\partial f}{\partial y} = \frac{1}{y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \quad = \quad \frac{\partial^2 f}{\partial y \partial x} = 0$$

Announcements

- Updated schedule
- This lecture will not be on Midterm 1.
  12.1 – 14.2
- Tomorrow is a Review (no participation)

Quick Review

Show the limit exist & calculate the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

$$\boxed{\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad xy = r^2\cos\theta\sin\theta \qquad \lim_{r\to 0} \frac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}} = 0}$$

when $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

when $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

when $y=mx$

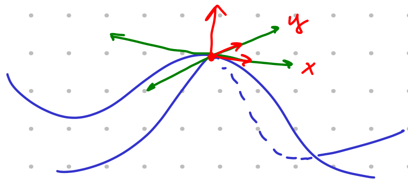
$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 \quad ✗$$

Section 14.4: Tangent Planes

- Suppose a function f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ is

$$\boxed{(z - z_0) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$$

* Why does this make sense?



$$\begin{vmatrix} i & j & k \\ 1, & 0, & f_x \\ 0, & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

$$\langle \vec{r} - \vec{r_0} \rangle \cdot \vec{n} = 0$$

- Example: Let $f(x,y) = 2x^2 + y^2$. Find the tangent plane to the graph of f at $(1,1,3)$

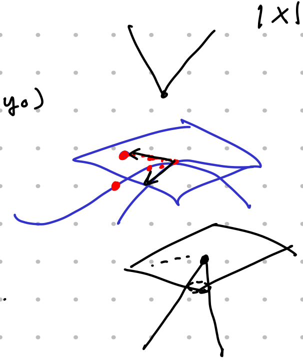
$f_x = 4x \qquad f_x(1,1) = 4$

$f_y = 2y \qquad f_y(1,1) = 2$

$$(z - 3) = 4(x-1) + 2(y-1)$$

☐ Linear approximations:

$|x|$



We can write any tangent plane $(z - z_0) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$ as a linear function of two variables:

$$\boxed{L_p(x,y) = Ax + By + C} \qquad (p = (x_0, y_0))$$

This is called the linear approximation of $f(x,y)$ at the point (x_0, y_0). It can also be called the linearization of $f(x,y)$ at (x_0, y_0).

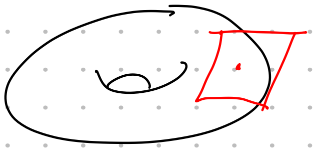
A function $f(x,y)$ is called differentiable at a point $p = (x_0, y_0)$ if the linearization $L_p(x,y)$ is a "good approximation" of the graph of $f(x,y)$.

It's easiest to consider the following theorem the definition of differentiable:

> **Theorem:** If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b) then $f(x,y)$ is differentiable at (a,b).

* This is the idea of a manifold.

"Locally Euclidean"



Section 14.5: The Chain Rule

Case 1: Suppose $z = f(x,y)$ is differentiable & $x = g(t)$, $y = h(t)$ are both differentiable. then:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$f(t) = \sin^2(t)\cdot e^{2t}$

$f(x,y) = x^2 \cdot y^2$
$x = \sin(t)$
$y = e^t$

$\frac{\partial f}{\partial x} = 2x\cdot y^2$ $\quad \frac{dx}{dt} = \cos(t)$

$\frac{\partial f}{\partial y} = 2x^2\cdot y$ $\quad \frac{dy}{dt} = e^t$

$\frac{df}{dt} = 2x\cdot y^2 \cdot \cos(t) + 2x^2\cdot y\, e^t$

Case 2: Suppose $z = f(x,y)$ is differentiable & $x = g(s,t)$, $y = h(s,t)$ are both differentiable.

then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$f(x,y) = x^2 \cdot y^2$
$x = \sin(t)s$
$y = e^t \cdot s^2$

$\frac{\partial f}{\partial x} = 2x\cdot y^2$ $\quad \frac{\partial x}{\partial s} = \sin(t)$

$\frac{\partial f}{\partial y} = 2x^2\cdot y$ $\quad \frac{\partial y}{\partial s} = 2e^t\cdot s$

$\frac{\partial f}{\partial s} = 2x\cdot y^2 \cdot \sin(t) + 2x^2\cdot y \cdot 2e^t\cdot s$

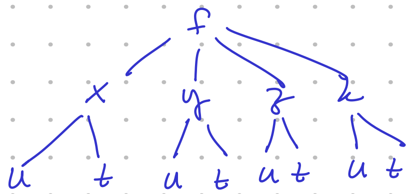
The General Case: Suppose u is a differentiable function of n variables $x_1, x_2, \dots, x_n$ and each x_i is a differentiable function of m variables $t_1, t_2, \dots, t_m$. then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

u = population of bunnies
(rainfall, # foxes, hunters)
(diseases) (traffic) (president)

Example: Write the chain rule for the case where $w = f(x,y,z,k)$ and $x = x(u,t)$, $y = y(u,t)$, $z = z(u,t)$, $k(u,t)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u} + \frac{\partial f}{\partial t}\cdot\frac{\partial k}{\partial u}$$



- Implicit Differentiation $\qquad x^2 + y^2 - 1 = 0 \qquad F(x,y) = x^2 + y^2 - 1$

Theorem: Suppose an equation $F(x,y) = 0$ defines y implicitly as a function of x. Then if F is differentiable,

$$\boxed{\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y}}$$

$F_x = 2x \qquad F_y = 2y$

$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

If z is given implicity as $z(x,y)$ by $F(x,y,z) = 0$, & F is differentiable,

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \& \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

* The proofs of these utilize the Implicit Function theorem which is beyond the scope of this class.

$F(x,y,z) = 0$ $\boxed{z(x,y)}$ Find : $\frac{\partial z}{\partial x}$

$$F_x \cdot \underbrace{\cancel{\frac{\partial x}{\partial x}}}_{1} + F_y \cdot \underbrace{\cancel{\frac{\partial y}{\partial x}}}_{0} + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$F_x + F_z \frac{\partial z}{\partial x} = 0 \xrightarrow{\text{solve for } \frac{\partial z}{\partial x}} F_z \frac{\partial z}{\partial x} = -F_x$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}}$$ as desired.

Examples + Practice Problems

Section 14.3

Find the first partials

19. $z = \ln(x + t^2)$

21. $f(x, y) = \dfrac{x}{y}$

23. $f(x, y) = \dfrac{ax + by}{cx + dy}$

25. $g(u, v) = (u^2 v - v^3)^5$

27. $R(p, q) = \tan^{-1}(pq^2)$

29. $F(x, y) = \int_y^x \cos(e^t)\, dt$

$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

27) $\dfrac{\partial R}{\partial p} = \dfrac{q^2}{1+(pq^2)^2}$

$\dfrac{\partial R}{\partial q} = \dfrac{2pq}{1+(pq^2)^2}$

21) $f_x = \dfrac{1}{y}$ $\quad f_y = \dfrac{-x}{y^2}$

29) $F(x,y) = \int_y^x \cos(e^t)\,dt$

$F(x,y) = \underbrace{\int_a^x \cos(e^t)\,dt}_{G(x)} - \underbrace{\int_a^y \cos(e^t)\,dt}_{H(y)}$

$\dfrac{\partial F}{\partial x} = \dfrac{dG}{dx} = \cos(e^x)$ $\quad \dfrac{\partial F}{\partial y} = \dfrac{dH}{dy} = \cos(e^y)$

19) $\dfrac{\partial z}{\partial t} = \dfrac{2t}{x - t^2}$ $\quad \dfrac{\partial z}{\partial x} = \dfrac{1}{x + t^2}$ $\quad \dfrac{\partial^2 z}{\partial x \partial t} = \dfrac{-2t}{(x+t^2)^2}$ $\quad \dfrac{\partial^2 z}{\partial t \partial x} = \dfrac{-(2t)}{(x+t^2)^2}$

Section 14.4

1–6 Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 2x^2 + y^2 - 5y$, $\quad (1, 2, -4)$

2. $z = (x + 2)^2 - 2(y - 1)^2 - 5$, $\quad (2, 3, 3)$

3. $z = e^{x-y}$, $\quad (2, 2, 1)$

4. $z = x/y^2$, $\quad (-4, 2, -1)$

5. $z = x \sin(x + y)$, $\quad (-1, 1, 0)$

6. $z = \ln(x - 2y)$, $\quad (3, 1, 0)$

5) $(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$f_x = x \cdot \cos(x+y) + \sin(x+y)$

$f_x(-1, 1) = (-1)\cdot\cos(0) + \sin(0) = -1$

$f_y = x\cos(x+y)$

$f_y(-1,1) = (-1)\cos(0) = -1$

$z = -(x+1) - (y-1)$

$\boxed{z = -x - y}$

17–18 Verify the linear approximation at (0, 0).

17. $\underbrace{e^x \cos(xy)}_{f(x,y)} \approx \boxed{x + 1}$ $\qquad$ **18.** $\dfrac{y - 1}{x + 1} \approx x + y - 1$

Point: $(0, 0, 1)$

$f_x = -ye^x \sin(xy) + e^x \cos(xy)$

$f_x(0,0) = 1$

$f_y = -xe^x \sin(xy)$

$f_y(0,0) = 0$

$(z - 1) = 1(x)$

$\boxed{z = x + 1}$

Plug in other points

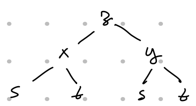
$(1, \pi)$

Section 14.5

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = (x - y)^5$, $\quad x = s^2 t$, $\quad y = st^2$

8. $z = \tan^{-1}(x^2 + y^2)$, $\quad x = s \ln t$, $\quad y = te^s$



$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}}$$

7) $\dfrac{\partial z}{\partial x} = 5(x-y)^4$ $\quad \dfrac{\partial x}{\partial s} = 2st$

$\dfrac{\partial z}{\partial y} = -5(x-y)^4$ $\quad \dfrac{\partial y}{\partial s} = t^2$

$$\boxed{\frac{\partial z}{\partial s} = 5(x-y)^4 \cdot 2st - 5(x-y)^4 \cdot t^2}$$

8) $\frac{d}{dx}\tan^{-1}(x) \quad \frac{1}{1+x^2}$

Find: $\frac{\partial z}{\partial t}$

$\frac{\partial z}{\partial y} = \frac{2y}{1+(x^2+y^2)^2}$ $\quad \frac{\partial z}{\partial x} = \frac{2x}{1+(x^2+y^2)^2}$

$\frac{\partial x}{\partial s} = \ln(t)$ $\quad \frac{\partial y}{\partial s} = t \cdot e^s$ $\quad \frac{\partial x}{\partial t} = \frac{s}{t}$ $\quad \frac{\partial y}{\partial t} = e^s$

27–30 Use Equation 6 to find dy/dx.

$$\frac{dy}{dx} = \frac{-F_x}{F_y} \qquad F(x,y)=0$$

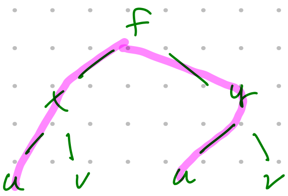
27. $y\cos x = x^2 + y^2$ **28.** $\cos(xy) = 1 + \sin y$

29. $\tan^{-1}(x^2y) = x + xy^2$ **30.** $e^y \sin x = x + xy$

27) $\underbrace{y\cos x - x^2 - y^2}_{F} = 0$

$$\left.\begin{aligned} F_x &= -y\sin(x) - 2x \\ F_y &= \cos x - 2y \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{y\sin(x) + 2x}{\cos x - 2y} \quad ✓$$

15. Suppose f is a differentiable function of x and y, and $g(u,v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0,0)$ and $g_v(0,0)$.



	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

$f(x(u,v), y(u,v))$

$$\boxed{\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}}$$

$x(u,v) = e^u + \sin v$

$\frac{\partial x}{\partial u} = e^u \qquad \frac{\partial x}{\partial u}(0,0) = 1$

$\frac{\partial y}{\partial u} = e^u \qquad \frac{\partial y}{\partial u}(0,0) = 1$

$g(u,v) = f(x(u,v), y(u,v))$

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial g}{\partial u}(u=0, v=0) = \frac{\partial f}{\partial u}(u=0, v=0)$$

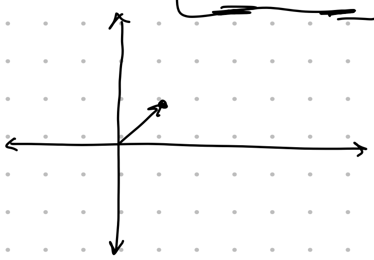
$$= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}(u=0, v=0) + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}(u=0, v=0)$$

$$= \underbrace{\frac{\partial f}{\partial x}(x=1, y=2)}_{2}\cdot\underbrace{\frac{\partial x}{\partial u}(u=0, v=0)}_{1}$$

$$+ \underbrace{\frac{\partial f}{\partial y}(x=1, y=2)}_{5}\cdot\underbrace{\frac{\partial y}{\partial u}(u=0, v=0)}_{1} = 7$$

45. If $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\partial z/\partial r$ and $\partial z/\partial\theta$ and (b) show that

$$\boxed{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$$

$$\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta\right)^2 + \left(\frac{\partial z}{\partial y}\sin\theta\right)^2 + 2\left(\frac{\partial z}{\partial x}\cos\theta\right)\left(\frac{\partial z}{\partial y}\sin\theta\right)$$

$$\left(-\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta\right) = \left(\frac{\partial z}{\partial x}\cdot r\sin\theta\right)^2 + \left(\frac{\partial z}{\partial y} r\cos\theta\right)^2 - 2\left(\frac{\partial z}{\partial x}\cdot r\sin\theta\right)\left(\frac{\partial z}{\partial y} r\cos\theta\right)$$

$$\left(\left(\frac{\partial z}{\partial x}\cos\theta\right)^2 + \left(\frac{\partial z}{\partial y}\sin\theta\right)^2 + 2\left(\frac{\partial z}{\partial x}\cos\theta\right)\left(\frac{\partial z}{\partial y}\sin\theta\right)\right) + \frac{1}{r^2}\left(\left(\frac{\partial z}{\partial x}^2\cdot r^2\sin^2\theta\right) + \left(\frac{\partial z}{\partial y}^2 r^2\cos^2\theta\right) - 2\left(\frac{\partial z}{\partial x}\cdot r\sin\theta\right)\left(\frac{\partial z}{\partial y} r\cos\theta\right)\right)$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

$$\parallel$$

$$z_x^2\sin^2\theta + z_y^2\cos^2\theta - 2z_x\sin\theta\cos\theta$$