Tuesday June 1, 2021
MTH 164 Lecture Notes

* look at both exams on the website.

$D \longrightarrow \mathbb{R}$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

Chapter 14: Partial Derivatives

Section 14.1: Functions of several variables

Definition: A function of n variables is a rule that assigns to each ordered n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ in a set $D \subset \mathbb{R}^n$, a real number denoted by $f(x_1, \dots, x_n)$. The set D is called the domain of f and its range is the set of values f takes:

$$\text{Range} = \{f(x_1, \dots, x_n) \in \mathbb{R} : (x_1, \dots, x_n) \in D\}$$

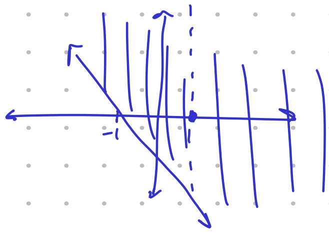
$y \geq -1 - x$

Example: For each function, evaluate $f(3,2)$ and find and sketch the domain

a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$

Domain: $\{(x,y) \in \mathbb{R}^2 : x \neq 1 \ \& \ x+y+1 \geq 0 \Rightarrow x+y \geq 1\}$

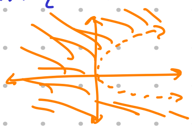
$f(3,2) = \dfrac{\sqrt{5+1}}{2} = \dfrac{\sqrt{6}}{2}$



b) $f(x,y) = x\ln(y^2 - x)$

Domain: $\{(x,y) \in \mathbb{R}^2 : (y^2 - x) > 0\}$

$y^2 > x$



$\ln(a) = y$

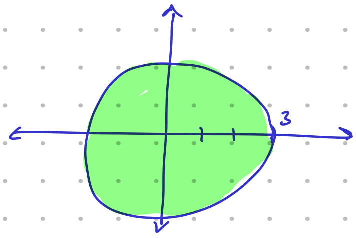
$a = e^y$

Example: Find the domain and range of $g(x,y) = \sqrt{9 - x^2 - y^2}$

Domain: $\{(x,y) \in \mathbb{R}^2 : 9 - x^2 - y^2 \geq 0 \Rightarrow 9 \geq x^2 + y^2\}$

Range: $\{k \in \mathbb{R} : \sqrt{9 - x^2 - y^2} = k, \ [0, 3]\}$

$9 = x^2 + y^2$



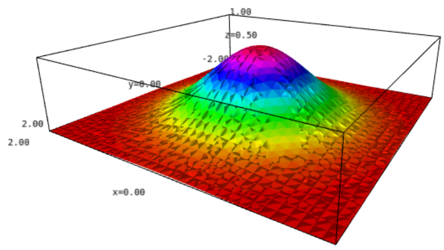
◻ Graphs

Definition: If f is a function of two variables with domain D, then the graph of f is

$$\text{graph}(f) = \{(x,y,z) \in \mathbb{R}^3 : z = f(x,y), \ (x,y) \in D\}$$

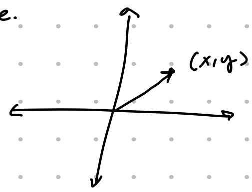
$f(x,y,u)$

Notice that the graph of a function of two variables is an example of a surface.



graph of $f(x,y) = e^{-x^2-y^2}$

radial symmetry e^{-r^2}



$f(x,y) = e^{-x^2-y^2}$ $\quad e^{-4}$ $\quad e^{-(.5)^2}$ $\quad .0183$

$e^{-x^2-y^2} = k$ $\quad .1353$

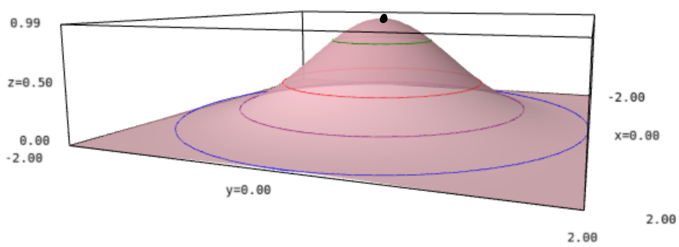
Let $x = \cos\theta$ $\quad r(\theta,R) = (R\cos\theta, R\sin\theta, e^{-R^2})$ $\quad 1.414$

$y = \sin\theta$

e $\quad e^{-R^2} = k$ $\quad e^{-4} = k$ $\quad k = \frac{1}{e^4}$ $\quad .7788$

◻ Level curves

Definition: The level curves of a function f a curves of the form $f(x,y) = k$ where k is a constant in the range of f. (Think traces).



some level curves on $f(x,y) = e^{-x^2-y^2}$

$e^{-x^2-y^2} = 1$ $\quad (0,0)$

let $x = r\cos\theta$ $\quad y = r\sin\theta$

e^{-r^2}

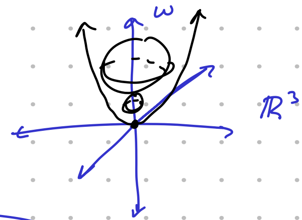
When we have 3 or more variables, we can no longer visualize the graphs but we can still get an idea of how the function behaves.

Example: Find the domain of $f(x,y,z) = \ln(z-y) + xy\sin z$

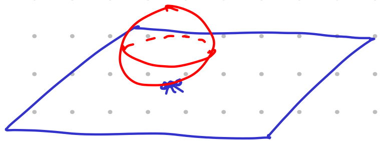
Domain: $\{(x,y,z) \in \mathbb{R}^3 : z-y \geq 0\}$

$w = z^2$

$w = x^2+y^2+z^2$



Example: Find the level surfaces of the function $\boxed{f(x,y,z) = x^2+y^2+z^2}$



$x^2+y^2+z^2 = 0 \rightsquigarrow (0,0,0)$

$x^2+y^2+z^2 = 1 \rightsquigarrow$ sphere

$x^2+y^2+z^2 = 2 \rightsquigarrow$ sphere

$x=0$	$y^2+z^2=1$
$y=0$	$x^2+z^2=1$
$z=0$	$y^2+x^2=1$

The level surfaces are spheres centered at the origin

✱ ✱ Section 14.2: Limits and Continuity ✱ ✱

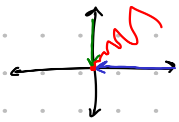
Example: Compare the following:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$

$$\lim_{r\to 0} \frac{\sin(r)}{r} = 1 \qquad r = x^2+y^2$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ DNE}$$

What can you say about how these functions behave at the origin?



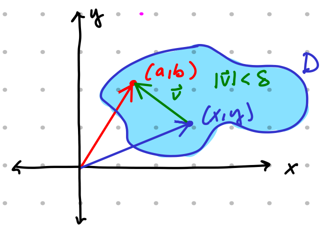
when $y=0$: $\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = \underline{1}$

when $x=0$: $\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = \underline{-1}$

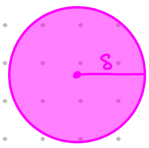
Definition: Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

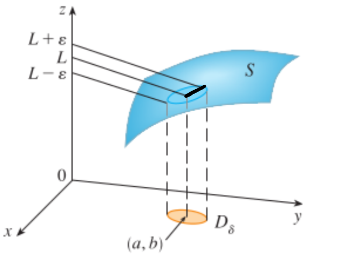
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.



Domain of $f(x,y)$





" The limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to L if I can make $f(x,y)$ as close to L as I want "

Important Ideas:

1. Going along <u>all possible</u> paths (not just straight ones).

Example: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

$x = 0$

$y = 0$

Example: What about $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$?

$$\lim_{(x,y)\to(0,0)} \frac{x\cdot y^3}{x^3+y^9}$$

$x = my$ $\quad \lim_{y\to 0} \frac{my^3}{m^2y^2+y^4} = \lim_{y\to 0} \frac{my}{m^2+y^2} = 0$

Let $x = y^2$ $\quad \lim_{y\to 0} \frac{\cancel{y^4}}{2\cdot \cancel{y^4}} = \frac{1}{2}$

2. Radial symmetry

Example: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Plug in $x^2+y^2 = r$

3. Brute force / Squeeze

Example: $\displaystyle\lim_{(x,y)\to(0,0)} \underbrace{\frac{3x^2y}{x^2+y^2}}_{\text{“}f(x,y)\text{”}}$

$$\boxed{\lim_{|\langle x,y\rangle|\to 0} f(x,y) \to 0 \text{ iff } \lim_{|\langle x,y\rangle|\to 0} |f(x,y)| \to 0}$$

Let $\varepsilon > 0$.
We want to find $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$ then $\left|\frac{3x^2y}{x^2+y^2}\right| < \varepsilon$

Hint: $x^2 \leq x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \leq 1$

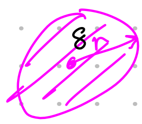
$\boxed{\text{Assume } \sqrt{x^2+y^2} < \delta}$ then $\Rightarrow$ $\left|\frac{3x^2y}{x^2+y^2}\right| < \varepsilon \iff \frac{3x^2|y|}{x^2+y^2} < \varepsilon$

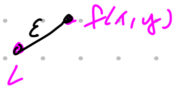
$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| \quad \text{write } \varepsilon \text{ in terms of } \delta$$
$$= 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2} \leq 3\cdot\delta = \varepsilon$$

the limit of $f(x,y)$ as (x,y) approaches (a,b) is L
and we write
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$
if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that
if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.

$\varepsilon = \frac{1}{3}\cdot\delta$

$\Rightarrow |f(x,y) - L| \leq \varepsilon$ $\square$





squeezing the range

$$\lim_{(x,y)\to(0,0)} \underbrace{\left|\frac{3x^2y}{x^2+y^2}\right|}_{<\varepsilon} = \lim_{(x,y)\to(0,0)} \left|3y\cdot\frac{x^2}{x^2+y^2}\right| \leq \lim_{(x,y)\to(0,0)} |3y| = 0$$

squeezing the range “δ”

$\Rightarrow$ Limit must be zero.

4. Squeeze : $\displaystyle\lim_{|\langle x,y\rangle|\to 0} f(x,y) \to 0$ iff $\displaystyle\lim_{|\langle x,y\rangle|\to 0} |f(x,y)| \to 0$

Example: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2}$

$$\frac{x^2}{2x^2+y^2} < 1 \qquad \left|\frac{x^2y^3}{2x^2+y^2}\right| \leq |y^3| \longrightarrow 0 \text{ as } y \to 0 \quad ✓$$

$$\frac{A}{B} \quad B > A,$$

☐ Continuity

Definition: A function of two variables is continuous at (a,b) if $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.
We say f is continuous on D if its continuous at all the points in D.

Example: Is the function continuous?

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Example: Where is $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous?

- $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\}$
- Everywhere except at $(x,y) = (0,0)$
- $\mathbb{R}^2 - \{(0,0)\}$

Examples & Practice Problems

Section 14.1

32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$ III

(b) $f(x, y) = \dfrac{1}{1 + x^2y^2}$ I

IV (c) $f(x, y) = \ln(x^2 + y^2)$

(d) $f(x, y) = \cos\sqrt{x^2 + y^2}$ V

VI (e) $f(x, y) = |xy|$

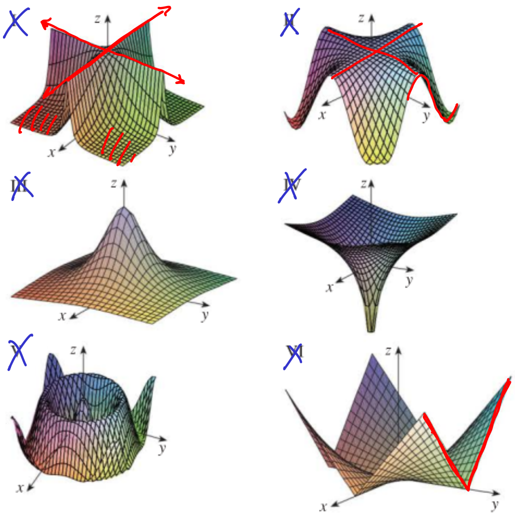
(f) $f(x, y) = \cos(xy)$ II

radially symmetric: only depends on x^2+y^2 = distance from origin.

$1 = \dfrac{1}{1+x^2y^2}$

$1 + x^2y^2 = 1$

$x^2y^2 = 0$



$\ln(a) = y$

$a = e^y$

e^{-500}

Range of ln can be negative

$z = f(x,y) = k$

67–70 Describe the level surfaces of the function.

67. $f(x, y, z) = x + 3y + 5z$

68. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

69. $f(x, y, z) = y^2 + z^2$

70. $f(x, y, z) = x^2 - y^2 - z^2$

67) $x + 3y + 5z = 0$ plane through the origin

$x + 3y + 5z = 1$

$x + 3y + 5z = k$

the level surfaces are all the planes with normal vector $\vec{n} = \langle 1, 3, 5 \rangle$

68) $x^2 + 3y^2 + 5z^2 = k$

$\dfrac{x^2}{k} + \dfrac{3y^2}{k} + \dfrac{5z^2}{k} = 1$ (concentric ellipsoids of the form $\dfrac{x^2}{k} + \dfrac{3y^2}{k} + \dfrac{5z^2}{k} = 1$)

69) $y^2 + z^2 = k$ cylinders w/ axis x & of the form $y^2 + z^2 = k$

Section 14.2

5–22 Find the limit, if it exists, or show that the limit does not exist.

5. $\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$

6. $\lim_{(x,y)\to(2,-1)} \dfrac{x^2y + xy^2}{x^2 - y^2}$

7. $\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x - y)$

8. $\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}}$

9. $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$

10. $\lim_{(x,y)\to(0,0)} \dfrac{5y^4\cos^2 x}{x^4 + y^4}$

11. $\lim_{(x,y)\to(0,0)} \dfrac{y^2\sin^2 x}{x^4 + y^4}$

12. $\lim_{(x,y)\to(1,0)} \dfrac{xy - y}{(x-1)^2 + y^2}$

13. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}}$

14. $\lim_{(x,y)\to(0,0)} \dfrac{x^3 - y^3}{x^2 + xy + y^2}$

15. $\lim_{(x,y)\to(0,0)} \dfrac{xy^2\cos y}{x^2 + y^4}$

16. $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^4 + y^4}$

17. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

18. $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^2 + y^8}$

19. $\lim_{(x,y,z)\to(\pi,0,1/3)} e^{y^2}\tan(xz)$

20. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz}{x^2 + y^2 + z^2}$

$\lim_{(x,y)\to(0,0)} \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$ No?

when $x = 0$

$\lim_{y\to 0} \dfrac{-4y^2}{2y^2} = -2$

when $y = x^2$

$\lim_{x\to 0} \dfrac{x^4 - 4x^4}{x^2 + 2x^4} = \lim_{x\to 0} \dfrac{-3x^2}{1 + 2x^2} = 0$

$0 \neq -2 \Rightarrow$ DNE

- polar
- "squeeze"
- going along diff paths
- continuous.

$\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

when $x = y$

$\lim_{x\to 0} \dfrac{x^2}{\sqrt{2x^2}} = \lim_{x\to 0} \dfrac{x^2}{\sqrt{2}\,x} = 0$

$\lim_{y\to 0} \dfrac{y}{\sqrt{1+y^2}} = 0$

$\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq |y|$

$\left|\dfrac{x}{\sqrt{x^2+y^2}}\right| < 1$

$\lim_{y\to 0} |y| = 0$

$\Rightarrow \lim = 0$ ✓

$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$

$\lim_{(r,\theta)\to(0,\theta)} r^2 \cdot \ln(r^2) = 0$

$\boxed{\lim_{x\to 0^+} x \cdot \ln(x)}$

$\lim_{(r,\theta)\to(0,\theta)} r\cos\theta = 0$

39–41 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

39. $\lim_{(x,y)\to(0,0)} \dfrac{x^3 + y^3}{x^2 + y^2}$

40. $\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$

41. $\lim_{(x,y)\to(0,0)} \dfrac{e^{-x^2-y^2} - 1}{x^2 + y^2}$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(r,\theta)\to(0,\theta)} f(r,\theta)$

$\lim_{(r,\theta)\to(0,\theta)} \dfrac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \dfrac{r^{\cancel{3}}(\cos^3\theta + \sin^3\theta)}{\cancel{r^2}}$

$\lim_{(r,\theta)\to(0,\theta)} r(\cos^3\theta + \sin^3\theta) = 0$

$\lim_{r\to 0} \dfrac{e^{-r} - 1}{r} = \lim_{r\to 0} \dfrac{-e^{-r}}{1} = -1$

29–38 Determine the set of points at which the function is continuous.

29. $F(x, y) = \dfrac{xy}{1 + e^{x-y}}$

30. $F(x, y) = \cos\sqrt{1 + x - y}$

31. $F(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$

32. $H(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

- denominator can't be zero
- Can't take square root of a negative number.