Thursday, May 27, 2021

MTH 164 Lecture Notes

Author → Do Carmo "Curves + Surfaces"

Chapter 13: Vector Functions

Section 13.1: Vector Functions and Space Curves

- We have already seen some vector functions.
- $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$r: \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \qquad 2t+5$

Limits and continuity

$$\boxed{\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle}$$

$\vec{r}(t)$ is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

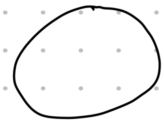
Example: $\vec{r}(t) = \left\langle \sin(t), t^2, \frac{t^2+1}{t} \right\rangle$

$$\lim_{t\to 0} \vec{r}(t) = \left\langle \lim_{t\to 0} \sin(t), \lim_{t\to 0} t^2, \lim_{t\to 0} \frac{t^2+1}{t} \right\rangle$$

$$= \langle 0, 0, DNE \rangle = DNE$$

Space Curves

A space curve is the trace of a continuous vector function. that is:

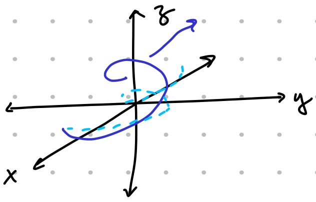


$$C = \{(x,y,z) : x = x(t), y = y(t), z = z(t)\}$$

Then, just like in the case of lines, the equations $x = x(t)$, $y = y(t)$ + $z = z(t)$ are called the parametric equations of C.

- Curves have both intrinsic and extrinsic properties. The intrinsic ones are those which do not depend on the parameterization (like curvature) and the extrinsic ones are those that do (like speed).

Example: $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Guess what this looks like!



Spiral

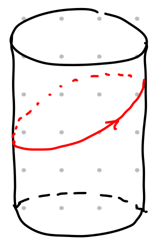
Example: Find a vector equation that describes the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.

Step I: Parameterize things!

$z = 2 - y$

$$\vec{r}(t) = \langle \cos(t), \sin(t), 2 - \sin(t) \rangle$$

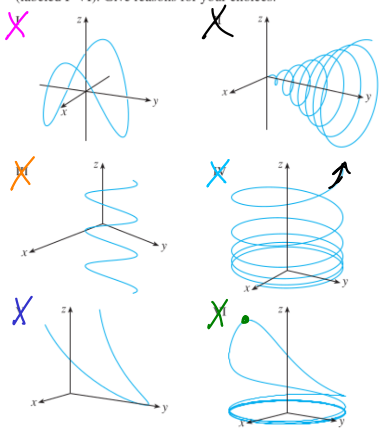
$$\vec{r}(t) = \langle \sin(t), \cos(t), 2 - \cos(t) \rangle$$



$x^2 + 5$

Matching Again

21–26 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.



21. $x = t\cos t$, $\boxed{y = t}$, $z = t\sin t$, $t \geq 0$ $\longleftrightarrow$ II

22. $x = \cos t$, $y = \sin t$, $z = 1/(1+t^2)$ $\longleftrightarrow$ VI

23. $x = t$, $y = 1/(1+t^2)$, $z = t^2$ $\leftarrow$ V

24. $x = \cos t$, $y = \sin t$, $z = \cos 2t$ $\leftarrow$ I

25. $x = \cos 8t$, $y = \sin 8t$, $z = e^{0.8t}$, $t \geq 0$ $\leftarrow$ IV

26. $x = \cos^2 t$, $y = \sin^2 t$, $z = t$ $\leftarrow$ III

$e^{-\frac{1}{2}} = \frac{1}{e^{1/2}}$ $\quad \frac{1}{e^t}$

$e^2 < e^3 < e^4$

- Projection.

  $\langle \cos(t), \sin(t) \rangle$

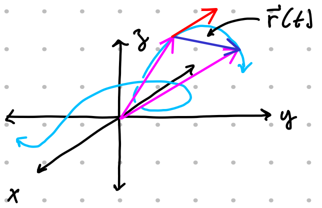
- Asymptotic Behavior

  what happens when $t \longrightarrow \pm\infty$?

$\lim_{t\to\infty} e^{.8t} = \infty$

$\lim_{t\to-\infty} e^{.8t} = 0$

Section 13.2: Derivatives + Integrals of Vector Functions



$\vec{r}(t) = \vec{r}(t+h) - \vec{r}(t)$

$$\boxed{\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle}$$

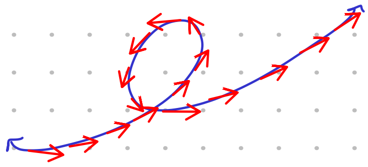
$\Rightarrow$ Tangent vectors

The unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Differentiation Rules

1. $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$
2. $\frac{d}{dt}(c\,\vec{u}(t)) = c\frac{d}{dt}\vec{u}(t)$
3. $\frac{d}{dt}(f(t)\cdot\vec{u}(t)) = \left(\frac{d}{dt}f(t)\right)\vec{u}(t) + f(t)\left(\frac{d}{dt}\vec{u}(t)\right)$
4. $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \left(\frac{d}{dt}\vec{u}(t)\right)\cdot\vec{v}(t) + \vec{u}(t)\cdot\left(\frac{d}{dt}\vec{v}(t)\right)$
5. $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \left(\frac{d}{dt}\vec{u}(t)\right)\times\vec{v}(t) + \vec{u}(t)\times\left(\frac{d}{dt}\vec{v}(t)\right)$
6. $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\,\vec{u}\,'(f(t))$ * Chain Rule.



$\cancel{\frac{d}{dt}f(t)\ \frac{d}{dt}u(f(t))}$

Example: Show that if $|\vec{r}(t)| = c$, then $\vec{r}\,'(t) \perp \vec{r}(t)\ \forall t$

$\vec{r}(t)\cdot\vec{r}(t) = c^2$

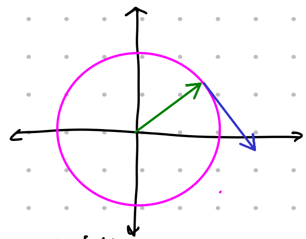
Show $\Rightarrow$ $\vec{r}\,'(t)\cdot\vec{r}(t) = 0$

$\frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t)) = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$

$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$

$= 2(\vec{r}(t)\cdot\vec{r}\,'(t)) = 0$

$\Rightarrow \vec{r}(t)\cdot\vec{r}\,'(t) = 0 \Rightarrow \vec{r}(t) \perp \vec{r}\,'(t)$ ✓



- Integrals

$$\boxed{\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\vec{i} + \left(\int_a^b y(t)\,dt\right)\vec{j} + \left(\int_a^b z(t)\,dt\right)\vec{k}}$$

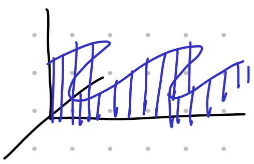
line integrals.

$\vec{r}(t) = \langle t, t^2, 2t \rangle$

$\int_0^1 \vec{r}(t)\,dt = \left\langle \int_0^1 t\,dt, \int_0^1 t^2\,dt, \int_0^1 2t\,dt \right\rangle$

$= \langle \frac{1}{2}, \frac{1}{3}, 1 \rangle$

$=$

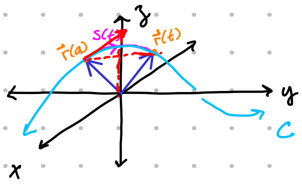


* What does it represent?

Section 13.3: Arc Length + Curvature

Back @ 10:11

- Arc length + curvature are both examples of intrinsic properties of curves.



$$s(t) = \int_a^t |\vec{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du$$

By the fundamental theorem of calculus: $\frac{ds}{dt} = |\vec{r}'(t)|$

- We can parameterize a curve with respect to arc length.

$\vec{r}'(t) = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1} = \sqrt{2}$

Example: $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ starting at $(1,0,0)$

$\frac{ds}{dt} = |\vec{r}'(t)| = \sqrt{2}$

$s(t) = \int_0^t \sqrt{2}\,du = \sqrt{2}\cdot t$

$s(t) = \sqrt{2}\,t$

$t(s) = \frac{1}{\sqrt{2}} \cdot s$

$\vec{r}(s) = \langle \cos\left(\frac{1}{\sqrt{2}} s\right), \sin\left(\frac{1}{\sqrt{2}} \cdot s\right), \frac{1}{\sqrt{2}} \cdot s \rangle$

Curvature

- A parameterization $\vec{r}(t)$ is called smooth if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$.
- A curve C is called smooth if it has a smooth parameterization.
- We only talk about the curvature of smooth curves.

$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

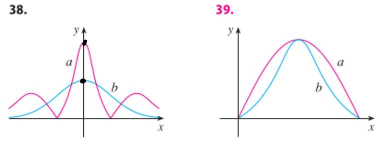
$\kappa = \text{Curvature} = \left|\frac{d\vec{T}}{ds}\right|$ where s is the arc length

$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} \Rightarrow \kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$

Theorem: $\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

38–39 Two graphs, a and b, are shown. One is a curve $y = f(x)$ and the other is the graph of its curvature function $y = \kappa(x)$. Identify each curve and explain your choices.

38. **39.**



$a = f(x)$

$b = \kappa(x)$

Example: Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}''(t) = \langle 0, 2, 6t \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = i(12t^2 - 6t^2) - j(6t) + k(2)$$

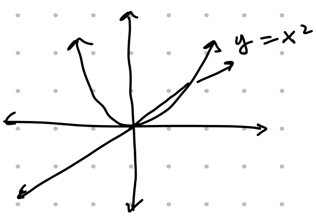
$= \langle 6t^2, -6t, 2 \rangle$

$|\vec{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$

$|\langle 6t^2, -6t, 2 \rangle| = \sqrt{36t^4 + 36t^2 + 4}$

$$\frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3}$$

Example: Show that, in the case of a graph of a function $y = f(x)$, $\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$



$\vec{r}(t) = \langle t, t^2, 0 \rangle$

$\vec{r}(t) = \langle t, f(t), 0 \rangle$

$\boxed{\vec{r}'(t) = \langle 1, f'(t), 0 \rangle}$

$\vec{r}''(t) = \langle 0, f''(t), 0 \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = \langle 0, 0, f''(t) \rangle$$

$|\vec{r}'(t)| = \sqrt{1 + (f'(t))^2} = (1 + f'(t))^{1/2}$

Examples & Practice Problems

Section 13.1

27. Show that the curve with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and ~~use this fact to help sketch the curve~~.

Show that the parametric equations satisfy the cone equation.

$$t^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\cos^2 t + \sin^2 t) = t^2 \quad \checkmark$$

28. Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

32. At what points does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

$$\sin^2 t + \cos^2 t + t^2 = 5$$
$$\Rightarrow 1 + t^2 = 5$$
$$\Rightarrow t^2 = 4$$
$$\Rightarrow t = \pm 2$$

At the points $\vec{r}(2)$ & $\vec{r}(-2)$

Answer

$P_1(\sin(2), \cos(2), 2)$

$P_2(\sin(-2), \cos(-2), -2)$

42–46 Find a vector function that represents the curve of intersection of the two surfaces.

42. The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$

43. The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$

44. The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$

45. The hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$

46. The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geq 0$, and the cylinder $x^2 + z^2 = 1$

43) $z = \sqrt{x^2 + y^2}$ $\quad z = 1 + y$

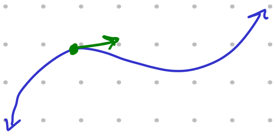
$$1 + y = \sqrt{x^2 + y^2}$$
$$(1 + y)^2 = x^2 + y^2$$
$$1 + 2y + \cancel{y^2} = x^2 + \cancel{y^2}$$
$$1 + 2y = x^2$$
$$y = \tfrac{1}{2}(x^2 - 1)$$

$$\vec{r}(t) = \left\langle t, \tfrac{1}{2}(t^2 - 1), 1 + \tfrac{1}{2}(t^2 - 1) \right\rangle$$

Section 13.2

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

23–26 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.



23. $x = t^2 + 1, \quad y = 4\sqrt{t}, \quad z = e^{t^2 - t}; \quad (2, 4, 1)$

24. $x = \ln(t + 1), \quad y = t \cos 2t, \quad z = 2^t; \quad (0, 0, 1)$

25. $x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad z = e^{-t}; \quad \underline{(1, 0, 1)}$

26. $x = \sqrt{t^2 + 3}, \quad y = \ln(t^2 + 3), \quad z = t; \quad (2, \ln 4, 1)$

$\vec{r}'(t) = \langle e^{-t}\sin(t) + -e^{-t}\cos(t), \; e^{-t}\cos(t) - e^{-t}\sin t, \; -e^{-t} \rangle$

$t=0 \quad \vec{r}'(0) = \langle -1, 1, -1 \rangle = \vec{v}$

$\vec{r}(t) = \langle 1-t, t, -1+t \rangle$

33. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

Calculate tangent vectors @ $(0,0,0)$

then find the angle between them.

35–40 Evaluate the integral.

35. $\int_0^2 (t\,\mathbf{i} - t^3\,\mathbf{j} + 3t^5\,\mathbf{k})\,dt$

36. $\int_1^4 (2t^{3/2}\,\mathbf{i} + (t + 1)\sqrt{t}\,\mathbf{k})\,dt$

Announcements

- Submit solutions to book problems to Gradscope if you want feedback
- Schedule will change a bit.
- Midterm 1 is a week from Monday (on 6/7)
    - 75 min.
    - Look out for an email.