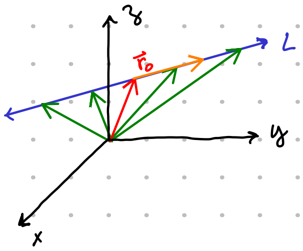
Wednesday, May 26, 2021

MTH 164 Lecture Notes

- Dot Products: $\langle a,b,c\rangle \cdot \langle u,v,w\rangle = au + bv + cw$
- Cross Products: $\langle a,b,c\rangle \times \langle u,v,w\rangle = \begin{vmatrix} i & j & k \\ a & b & c \\ u & v & w \end{vmatrix}$

Section 12.5: Equations of lines and planes

Lines

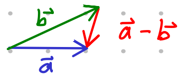


$\vec{r} = \langle x,y,z\rangle$ $\qquad$ $\langle x_0, y_0, z_0\rangle$

$\boxed{\text{The vector equation: } \vec{r} = \vec{r_0} + t\vec{v}}$ — direction vector $\langle a,b,c\rangle$

$\underline{t}$ is called the **parameter** and you should think of it as time.

Remember vector subtraction:



Let's write this out: $\langle x,y,z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle$

$$= \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

This gives us the **parametric** equations for the line:

$$x(t) = x_0 + ta \quad y(t) = y_0 + tb \quad z(t) = z_0 + tc$$

- The vector $\vec{v}$ is called the direction vector. Its components $a, b,$ and c are called direction numbers.

$\frac{x - x_0}{a} = t$

Solving all the parametric equations for t and setting them equal to each other gives us the **symmetric equations:**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example: Find a vector equation for the line that passes through the point $(5,1,3)$ and is parallel to $\langle 1,4,-2\rangle$

- $\vec{r} = \vec{r_0} + t\vec{v}$

$\vec{r_0} = \langle 5,1,3\rangle$
$\vec{v} = \langle 1,4,-2\rangle$

$\langle x,y,z\rangle = \langle 5,1,3\rangle + \langle t, 4t, -2\rangle$
$= \langle 5+t, 1+4t, 3-2t\rangle$

vector

parametric

$x(t) = 5 + t$
$y(t) = 1 + 4t$
$z(t) = 3 - 2t$

Example:

a) Find a vector equation for the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$

b) At what point does this line pass through the xy-plane?

a) $\vec{r_0} = \langle 2,4,-3\rangle$ $\qquad \vec{AB} = \langle 3-2, -1-4, 1+3\rangle = \langle 1,-5,4\rangle = \vec{v}$

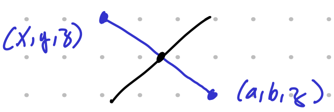
$\langle x,y,z\rangle = \langle 2+t, 4-5t, -3+4t\rangle$ — $z(t)$

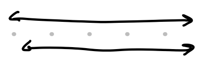
b) $-3+4t = 0$ solve for t: $t = \frac{3}{4}$ then plugin: $\vec{r}(3/4) = \langle 2 + \frac{3}{4}, 4 - \frac{15}{4}, 0\rangle$

$\Rightarrow$ Point: $\left(2 + \frac{3}{4}, 4 - \frac{15}{4}, 0\right)$

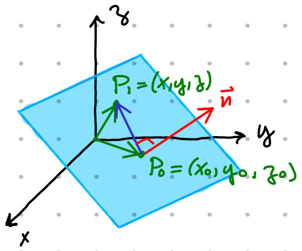
In general, the line segment connecting the positions $\vec{r_0}$ and $\vec{r_1}$ is $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$.

- Intersecting lines
- parallel lines
- skew lines.





Planes



$\langle x, y, z \rangle$

The vector equation for a plane is: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$

$$\langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$
$$= a(x - x_0) + b(y - y_0) + c(z - z_0)$$

This gives us a scalar equation for the plane that passes through the point (x_0, y_0, z_0) with normal vector $\vec{n}$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

* Note that this means any equation of the form $Ax + By + Cz + D = 0$ describes a plane.

Example: Find the equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$

$\vec{n} = \vec{PQ} \times \vec{RP}$

$$\vec{n} = \begin{vmatrix} i & j & k \\ \langle 1, & -2, & 2 \rangle \\ \langle -4, & 1, & 2 \rangle \end{vmatrix} = i(-4-2) - j(2+8) + k(1-8) = \begin{array}{c} a \quad b \quad c \\ \langle -6, -10, -7 \rangle \end{array}$$

$$\langle -6, -10, -7 \rangle \cdot \langle x-1, y-3, z-2 \rangle = 0$$
$$-6(x-1) - 10(y-3) - 7(z-2) = 0$$

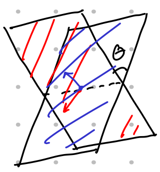
Example:

$\vec{n_1} = \langle 1, 1, 1 \rangle$ $\quad$ $\vec{n_2} = \langle 1, -2, 3 \rangle$

a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$

b) What is the equation of the line where they intersect?

$\langle x, y, z \rangle = \langle 0, 0, 1 \rangle + t\langle 0, 0, 1 \rangle$



a) angle between $\vec{n_1}$ + $\vec{n_2}$ $\quad \cos(\theta) = \dfrac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}| \cdot |\vec{n_2}|}$

b) $\vec{r} = \vec{r_0} + t\vec{v}$

$\vec{v} = \vec{n_1} \times \vec{n_2}$

$\vec{r_0} = (1, 0, 0)$

$x + y + z = 1$
$x - 2y + 3z = 1$ $\quad x = 1 \rightsquigarrow$ $2y + 2z = 0$
$-2y + 3z = 0$

$5z = 0 \Rightarrow z = 0 \Rightarrow y = 0$

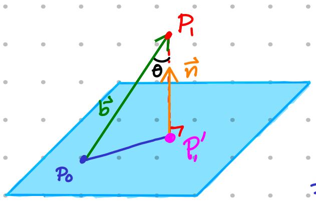
Distance

Example: Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

$\text{comp}_{\vec{b}}\, \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

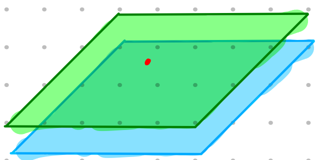
Use Projections:



$$\left|\text{comp}_{\vec{n}}\, \vec{P_0P_1}\right| = \frac{|\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \cdot \langle a, b, c \rangle|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example: Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$



distance from $P_1(0, 0, -1)$ to $10x + 2y - 2z = 5$

Example:

a) Show that the lines L_1 + L_2 are skew: not parallel + not intersecting

$L_1 = \langle 1+t, -2+3t, 4-t \rangle$ $\vec{v}_1 = \langle 1, 3, -1 \rangle$

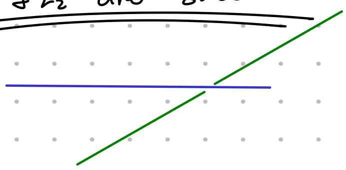
$L_2 = \langle 2s, 3+s, -3+4s \rangle$ $\vec{v}_2 = \langle 2, 1, 4 \rangle$

5 min 10:15

b) What is the distance between them?

* Hint: The distance is the same as the distance between the two parallel planes containing L_1 + L_2.

$\Rightarrow L_1$ + L_2 are skew.



$1+t = 2s$
$-2+3t = 3+s$

$\Rightarrow$ $-3-3t = -6s$
$-2+3t = 3+s$

$-5 = 3-5s$

$-8 = -5s \Rightarrow \boxed{s = 8/5}$

$1+t = 2 \cdot \frac{5}{8} = \frac{5}{4}$

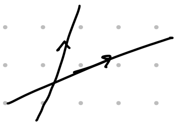
$t = \frac{5}{4} - \frac{4}{4} = \frac{1}{4}$

$t = \frac{1}{4}$

$-4 + 4 \cdot \frac{8}{5} \neq \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

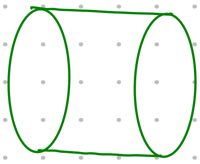
$\Rightarrow$ skew

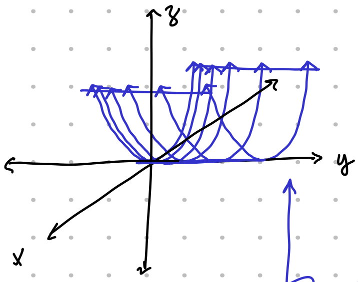
b)

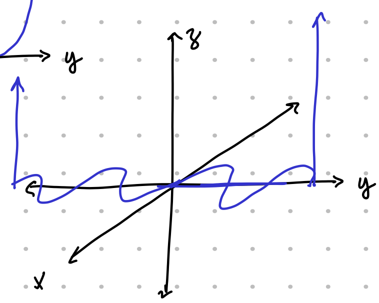


Section 12.6: Cylinders + Quadric Surfaces

- Cylinders
  - $x^2 + y^2 = 1$
  - $x^2 + z^2 = 1$
  - $z = x^2$
  - $y = \sin(x)$







- Quadric surfaces

A quadric surface is one of the form:

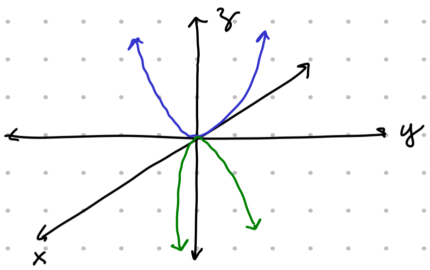
$$S = \{(x,y,z) \in \mathbb{R}^3 : Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0\}$$

Example: Use traces to sketch

- $z = 4x^2 + y^2$

When $x = 0$, $z = y^2$

when $y = 0$
$z = 4x^2$



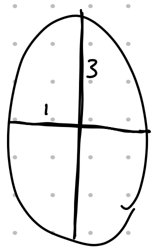
$x = 0$

$\frac{y^2}{9} + \frac{z^2}{4} = -1$

$-\frac{y^2}{9} - \frac{z^2}{4} = 1$

- $z = y^2 - x^2$
- $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$

- $x^2 - \frac{y^2}{9} - \frac{z^2}{4} = 1$



$z = y^2 - x^2$

$z = y^2$

$z = -x^2$

Let $z = 0$ $x^2 + \frac{y^2}{9} = 1$

ellipsoid

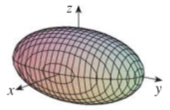
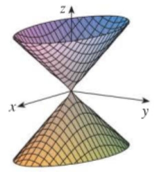
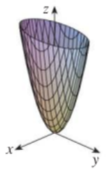
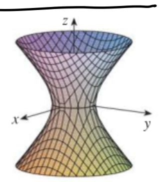
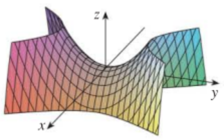
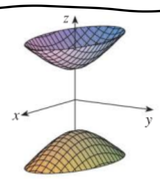
$x =$

$\frac{y^2}{9} - \frac{z^2}{4} = 1$

$z = 0$

$x^2 + \frac{y^2}{9} = 1$

Quadric surfaces are completely classified.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Some more examples
Classify the equations:

- $4x^2-y^2+2z^2+4=0$
- $x^2+2z^2-6x-y+10=0$

$$\frac{4x^2}{-4}+\frac{2z^2}{-4}-\frac{y^2}{-4}=1$$

$$-x^2-\frac{z^2}{2}+\frac{y^2}{4}=1$$

$$x^2-6x-y+2z^2=-10$$

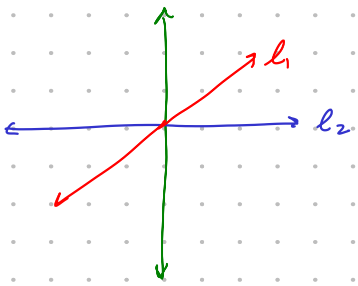
$$x^2-6x+2z^2=-10+y$$

$$(x-3)^2+2z^2=-1+y$$

Additional Practice Problems

Section 12.4

1. Determine whether each statement is true or false in $\mathbb{R}^3$.
(a) Two lines parallel to a third line are parallel. ✔
(b) Two lines perpendicular to a third line are parallel. ✘
(c) Two planes parallel to a third plane are parallel. ✔
(d) Two planes perpendicular to a third plane are parallel. ✘
(e) Two lines parallel to a plane are parallel. ✘
(f) Two lines perpendicular to a plane are parallel. ✔
(g) Two planes parallel to a line are parallel. ✘
(h) Two planes perpendicular to a line are parallel. ✔
(i) Two planes either intersect or are parallel. ✔
(j) Two lines either intersect or are parallel. ✘
(k) A plane and a line either intersect or are parallel. ✔



Write the equation of:

4. The line through the point $(0, 14, -10)$ and parallel to the line $x=-1+2t$, $y=6-3t$, $z=3+9t$

5. The line through the point $(1, 0, 6)$ and perpendicular to the plane $x+3y+z=5$

23. The plane through the origin and perpendicular to the vector $\langle 1, -2, 5\rangle$

24. The plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i}+\mathbf{j}-\mathbf{k}$

30. The plane that contains the line $x=1+t$, $y=2-t$, $z=4-3t$ and is parallel to the plane $5x+2y+z=1$

31. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$

48. Where does the line through $(-3, 1, 0)$ and $(-1, 5, 6)$ intersect the plane $2x + y - z = -2$?

67. Which of the following four planes are parallel? Are any of them identical?

P_1: $3x + 6y - 3z = 6$ $\qquad$ P_2: $4x - 12y + 8z = 5$

P_3: $9y = 1 + 3x + 6z$ $\qquad$ P_4: $z = x + 2y - 2$

68. Which of the following four lines are parallel? Are any of them identical?

L_1: $x = 1 + 6t, \quad y = 1 - 3t, \quad z = 12t + 5$

L_2: $x = 1 + 2t, \quad y = t, \quad z = 1 + 4t$

L_3: $2x - 2 = 4 - 4y = z + 1$

L_4: $\mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 4, 2, 8 \rangle$

69–70 Use the formula in Exercise 12.4.45 to find the distance from the point to the given line.

69. $(4, 1, -2)$; $\quad x = 1 + t, \quad y = 3 - 2t, \quad z = 4 - 3t$

70. $(0, 1, 3)$; $\quad x = 2t, \quad y = 6 - 2t, \quad z = 3 + t$

78. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

Section 12.6

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

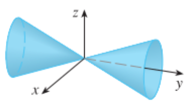
21. $x^2 + 4y^2 + 9z^2 = 1$ $\qquad$ **22.** $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$ $\qquad$ **24.** $-x^2 + y^2 - z^2 = 1$

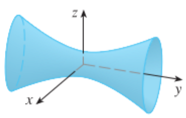
25. $y = 2x^2 + z^2$ $\qquad$ **26.** $y^2 = x^2 + 2z^2$

27. $x^2 + 2z^2 = 1$ $\qquad$ **28.** $y = x^2 - z^2$

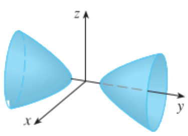
I



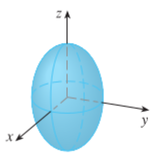
II



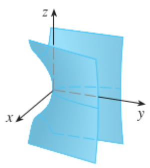
III



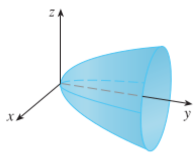
IV



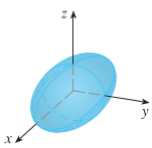
V



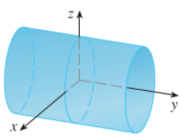
VI



VII



VIII



45. Find an equation for the surface obtained by rotating the curve $y = \sqrt{x}$ about the x-axis.

46. Find an equation for the surface obtained by rotating the line $z = 2y$ about the z-axis.

47. Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.

48. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.