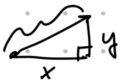
Tuesday, May 25, 2021

MTH 164 Lecture Notes

Section 12.3: The dot product

Definition: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the dot product of $\vec{b}$ and $\vec{a}$ is the <u>number</u> $\vec{a} \cdot \vec{b}$ given by

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$



(Similarly in n dimensions: $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$)

$$|\langle x, y \rangle| = \sqrt{x^2 + y^2}$$

Properties of the dot product

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
3. $\vec{0} \cdot \vec{a} = 0$
4. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
5. $(c\vec{a}) \cdot b = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

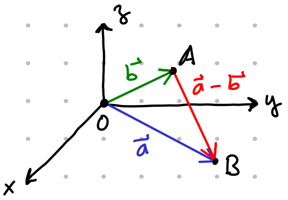
$$\begin{aligned} \langle x, y \rangle \cdot \langle x, y \rangle &= x \cdot x + y \cdot y \\ &= x^2 + y^2 \\ &= |\langle x, y \rangle|^2 \end{aligned}$$

Theorem: If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

You can think of this as the "geometric definition"

Proof:



- The Law of Cosines: $|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos\theta$
- Convert to vector notation: $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos\theta$
- Observation: $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$

$$\begin{aligned} &= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \\ &= |a|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \end{aligned}$$

- $\Rightarrow$ $|a|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos\theta$

$$\Rightarrow \quad -2\vec{a} \cdot \vec{b} = -2|\vec{a}||\vec{b}| \cos\theta \quad \Rightarrow \quad \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

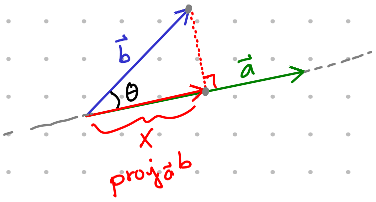
☐

Corollary: If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

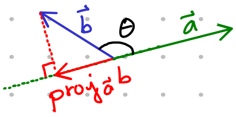
$\Rightarrow$ In particular two vectors $\vec{a}$ + $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$ since in this case, $\cos\theta = \cos(\pi/2) = 0$

☐ Projections



Vector projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|a|^2} \vec{a}$

Scalar projection of $\vec{b}$ onto $\vec{a}$: $\text{comp}_{\vec{a}} \vec{b} = \pm |\text{proj}_{\vec{a}} \vec{b}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$



* Why do these expressions describe the pictures at the left?

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}))$$

$$\text{proj}_{\vec{a}} \vec{b} = A \frac{\vec{a}}{|\vec{a}|} \qquad |\vec{b}| \cos(\theta) = \frac{|\vec{b}| \cdot \cancel{|\vec{a}|} \cdot \cos\theta}{\cancel{|\vec{a}|}} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

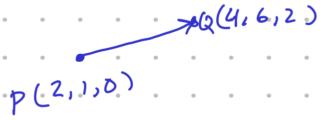
$$\text{proj}_{\vec{a}} \vec{b} = \underline{\underline{|\vec{b}| \cos(\theta)}} \cdot \frac{\vec{a}}{|\vec{a}|} \qquad \frac{\vec{a} \cdot \vec{b}}{|a|^2} \vec{a} = \frac{\cancel{|\vec{a}|} \cdot |\vec{b}| \cdot \cos\theta}{|\vec{a}|^{\cancel{2}}} \cdot \vec{a}$$

☐ Work & Force

The work done by a constant force $\vec{F}$ along a displacement vector $\vec{D}$ is

$$W = \vec{F} \cdot \vec{D}$$

Example: A force given by $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ moves a particle along a straight line from point $P(2,1,0)$ to point $Q(4,6,2)$. Find the work done.



$$\vec{D} = \vec{PQ} = \langle 4-2, 6-1, 2 \rangle = \langle 2,5,2 \rangle$$

$$\vec{F} \cdot \vec{D} = \langle 3,4,5 \rangle \cdot \langle 2,5,2 \rangle = 6+20+10 = 36$$

Section 12.4: The cross product

Definition: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the cross product of $\vec{a}$ and $\vec{b}$ is the vector

determinant

$$\boxed{\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2 , a_3b_1 - a_1b_3 , a_1b_2 - a_2b_1 \rangle}$$

*3Blue1Brown series
on linear algebra
Ep 6. Determinants
on my website 164 page

Matrix
Matrix mult.
determinant.

2 x 2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$$

$$\langle 1,0,0 \rangle$$

$$\langle a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), (a_1b_2 - a_2b_1) \rangle$$

$$-a_1b_3 + a_3b_1$$

* Unlike dot products, there is no generalization of the cross product in n dimensions.

Theorem: The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cdot \cos(\theta)$$
$$= 0 \iff \vec{u} \perp \vec{v}$$

Proof:

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3$$
$$= 0$$

Similarly for $(\vec{a} \times \vec{b}) \cdot \vec{b}$. ☐

You can think of this as the "geometric definition"

Theorem: If the angle θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

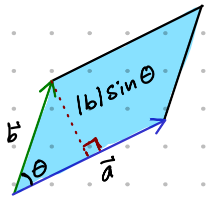
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cdot \cos(\theta)$$

Proof sketch:

- Write $|\vec{a} \times \vec{b}|^2$ in components. You will see:

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$
$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$
$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$
$$= |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta)$$
$$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$$
$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \text{ as desired.}$$

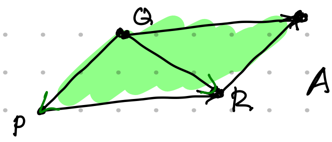


☐

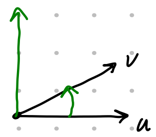
$$|\vec{a}||\vec{b}| \sin\theta$$

Corollary: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$.

Example: Find the area of a triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$



$$\text{Area} = \frac{|\vec{PQ} \times \vec{PR}|}{2}$$



$a \cdot b$
$2 \cdot 4 = 4 \cdot 2$

$u \times v$
$|v \times u| = |-u \times v|$

Properties of Cross Products

1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ anti-commutativity

2. $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times c\vec{b}$

3. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

4. $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$

left + right distributivity

5. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ associativity.

6. $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\underline{\vec{a}\cdot\vec{c}}) - \vec{c}(\underline{\vec{a}\cdot\vec{b}})$ "Bac Cab"

$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$2\vec{u}\times\vec{v} = \vec{u}\times 2\vec{v} = 2(\vec{u}\times\vec{v})$ ✓

49) Prove $(a-b)\times(a+b) = 2(a\times b)$

$(a+(-b))\times(a+b)$

$= a\times(a+b) + (-b)\times(a+b)$

$= \cancel{a\times a} + a\times b - b\times a - \cancel{b\times b}$

$= a\times b - b\times a$

$= a\times b + a\times b = 2(a\times b)$

☐ Triple Products

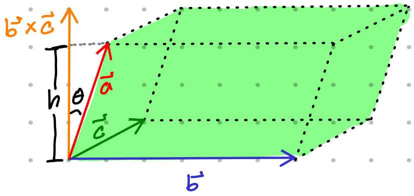
The product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product.

If you write it in components, you will notice that: $\vec{a}\cdot(\underset{\vec{v}}{\vec{b}\times\vec{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Also, using the geometric definition of the dot product:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = |\vec{a}|\cdot|\vec{b}\times\vec{c}|\cos\theta$$

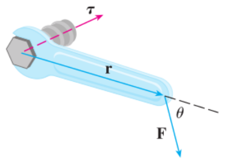
Volume $= |\vec{b}\times\vec{c}|\cdot|\vec{a}|\cos(\theta) = |\vec{a}\cdot(\vec{b}\times\vec{c})|$



=> The volume of the parallelpiped "spanned" by 3 vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the absolute value of the scalar triple product

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

☐ Torque



Torque $= \vec{\tau} = \vec{r}\times\vec{F}$

* See the 3Blue1Brown linear algebra series.

* Incidentally this also explains orientations. (episode 6)

Break: Back at 10:15

Examples + Practice Problems

Section 12.3

1. Which of the following expressions are meaningful? Which are meaningless? Explain.

(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ ✗
(b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ ✓
(c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ ✓
(d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ ✓
(e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ ✗
(f) $|\mathbf{a}| \bullet (\mathbf{b} + \mathbf{c})$ ✗

- Find the angle between $\vec{a} = \langle 4,3 \rangle$ and $\vec{b} = \langle 2,-1 \rangle$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

$$8 - 3 = \sqrt{16+9} \cdot \sqrt{5} \cdot \cos\theta$$

$$5 = 5\sqrt{5} \cos\theta$$

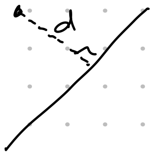
$$\cos\theta = \frac{1}{\sqrt{5}}$$

$$\theta = \cos^{-1}\left(1/\sqrt{5}\right)$$

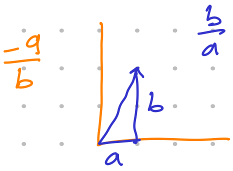
53. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

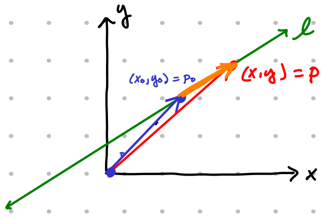
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.



Step 1: A line is given by $ax+by+c=0$. Let (x_0,y_0) be a point on our line.



Step 2: Write the equation of the line in vector notation.



Notice $ax+by+c=0 \iff by = -ax - c \iff y = \frac{-a}{b}x - c$

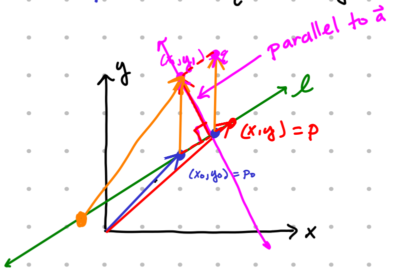
thus $\overrightarrow{P_0P} = \langle x,y \rangle - \langle x_0,y_0 \rangle = \langle x-x_0, y-y_0 \rangle$ must be orthogonal to $\langle a,b \rangle$

*Why?

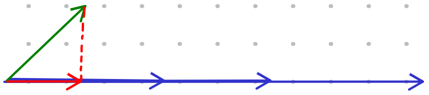
$\Rightarrow$ Another equation for the line:

$$\left\{ (x,y) \in \mathbb{R}^2 : \langle a,b \rangle \cdot \langle x-x_0, y-y_0 \rangle = 0 \right\}$$

Step 3: Let $q = (x_1, y_1)$ be an arbitrary point in $\mathbb{R}^2$.



Let $\vec{v} = \overrightarrow{P_0q} = \langle x_1 - x_0, y_1 - y_0 \rangle$



Step 4: Write distance as a projection. Simplify.

$$|\text{comp}_{\langle a,b \rangle} \vec{v}|$$

Vector projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|a|^2} \vec{a}$

Scalar projection of $\vec{b}$ onto $\vec{a}$: $\text{comp}_{\vec{a}}\vec{b} = \pm|\text{proj}_{\vec{a}}\vec{b}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$$ax_0 + by_0 = -c$$

$$ax + by + c = 0$$

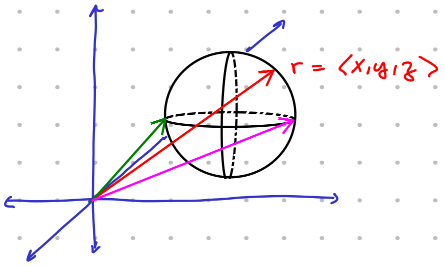
$$\langle x_1 - x_0, y_1 - y_0 \rangle \cdot \langle a,b \rangle = \frac{a(x_1 - x_0] + b\langle y_1 - y_0 \rangle}{\sqrt{a^2+b^2}} =$$

$$\frac{ax_1 - ax_0 + by_1 - by_0}{\sqrt{a^2+b^2}} = \frac{ax_1 + by_1 - (ax_0 + by_0)}{\sqrt{a^2+b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

$\square$ QED

variables

54. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.



Section 12.4

EXAMPLE 5 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

$$\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(-18+36) - 4(36) - 7(-18)$$

$$= 18 - 8(18) + 7(18) = 0 \quad ✓$$

$$\begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = (-18+36)i - (36)j + (-18)k$$

$$= \langle 18, -36, -18 \rangle$$

$\vec{a} \cdot (\vec{b} \times \vec{c})$

$\langle 1, 4, -7 \rangle \cdot \langle 18, -36, -18 \rangle$

$= 18 - 4 \cdot 36 + 7 \cdot 18$

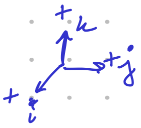
$18 - 8 \cdot 18 + 7 \cdot 18$

$= 18(1 - 8 + 7) = 0$

9–12 Find the vector, not with determinants, but by using properties of cross products.

9. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$ **10.** $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

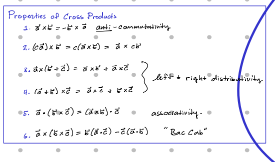
11. $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$ **12.** $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$



9) $-k \times (i \times j) = i\underbrace{(-k \cdot j)}_{0} + j\underbrace{(k \cdot i)}_{0} = 0$

10) $k \times i - k \times 2j$

$j - 2\underbrace{(k \times j)}_{-i} = j + 2i$



Properties of Cross Products

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ anti-commutativity
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times c\vec{b}$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ } left + right distributivity
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ }
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ associativity.
6. $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ "Bac Cab"

$$k \times j = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -i$$

$=$

13. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ (b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (d) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

(e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ (f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

44. (a) Find all vectors $\mathbf{v}$ such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$

(b) Explain why there is no vector $\mathbf{v}$ such that

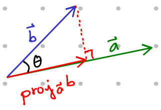
$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

46. (a) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

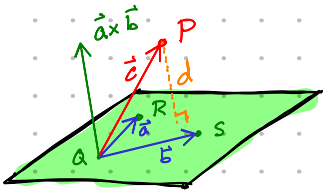
where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.



Vector projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|a|^2}\vec{a}$

Scalar projection of $\vec{b}$ onto $\vec{a}$: $\text{comp}_{\vec{a}}\vec{b} = \pm|\text{proj}_{\vec{a}}\vec{b}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$



49. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.