Monday, May 24, 2021

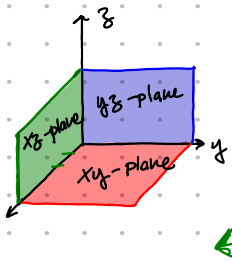
MTH 164 Lecture Notes

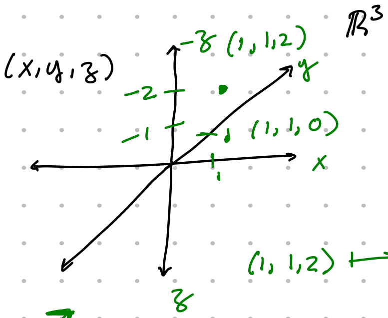
$\mathbb{R}^5$

Chapter 12: Vectors and the geometry of space

Section 12.1: Three-dimensional coordinate systems

- 3d-coordinates
- coordinate planes
- octants
- projections
- right-hand rule



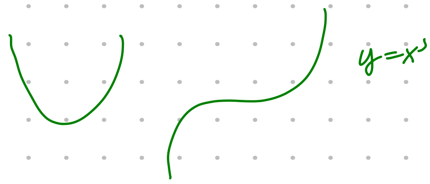


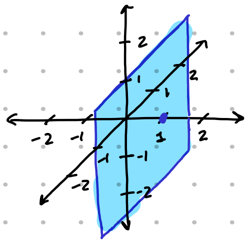
$(1,1,2) \mapsto$ xy-plane

"orientation"

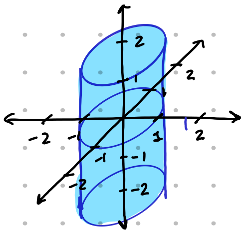
- Surfaces
  - An equation of $x, y,$ and z defines a surface in $\mathbb{R}^3$

Examples



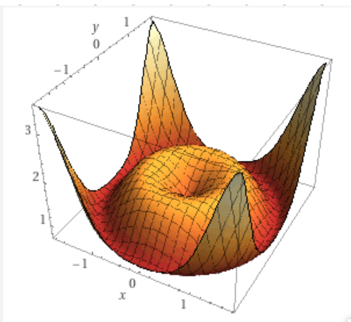


A plane $y = 1$

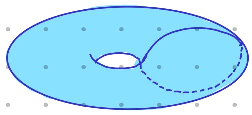


A cylinder $x^2+y^2=1$

$\vec{f}(r,s) = \langle x(r,s), y(r,s), z(r,s) \rangle$

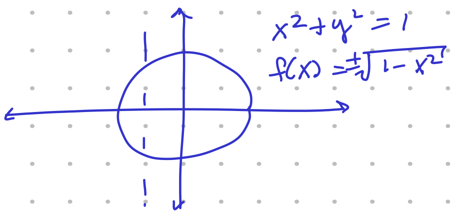


$z = \sqrt{x^2+y^2} + 3\cos(x^2+y^2)$



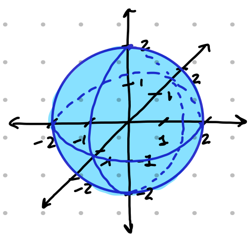
A torus

* More about surfaces later!



- Distances and spheres
  - $P_1 = (x_1, y_1, z_1)$
    $P_2 = (x_2, y_2, z_2)$

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$



- The equation of a sphere with center $C(h,k,\ell)$ and radius r:

$$\boxed{(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2}$$

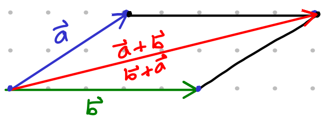
- When the center $= (0,0,0)$, we have $x^2+y^2+z^2=r^2$

Section 12.2: Vectors

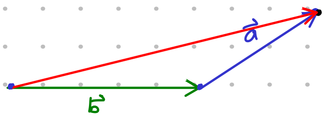
- Force
- velocity
- acceleration
- torque
- elasticity ?
- wind currents

- A vector is a quantity that has both a magnitude (length) and a direction
- The zero vector however, has no direction and length zero.
- What are some examples of vectors?

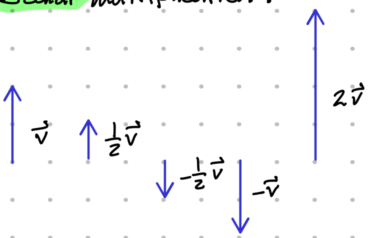
Addition:



So if $\vec{a} = \langle x, y, z\rangle$
and $\vec{b} = \langle u, v, w\rangle$
then $\vec{a}+\vec{b} = \langle x+u,\ y+v,\ z+w\rangle$

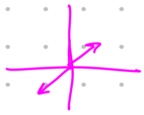


Scalar multiplication:

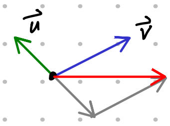


$\boxed{\begin{array}{l}\text{So for } c\in\mathbb{R},\ \vec{v} = \langle x,y,z\rangle \\ c\vec{v} = c\langle x,y,z\rangle = \langle cx, cy, cz\rangle\end{array}}$

$-1\cdot\langle 1,1\rangle = \langle -1,-1\rangle$

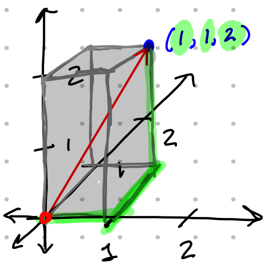


Subtraction:

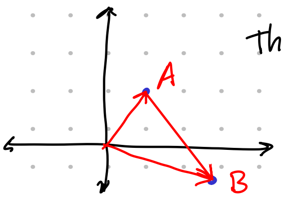


$\vec{v}-\vec{u} = \vec{v}+(-\vec{u})$

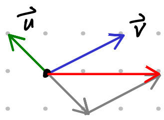
Components



- The vector starting at $A(x_1, y_1, z_1)$ and ending at $B(x_2, y_2, z_2)$ is denoted by $\overrightarrow{AB}$.
- Furthermore, $\overrightarrow{AB} = \vec{B}-\vec{A} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1\rangle$

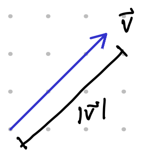


This picture again:



$\vec{v}-\vec{u} = \vec{v}+(-\vec{u})$

Magnitude



The length of a vector $\vec{v} = \langle x, y, z\rangle$ is $|\vec{v}| = \sqrt{x^2+y^2+z^2}$

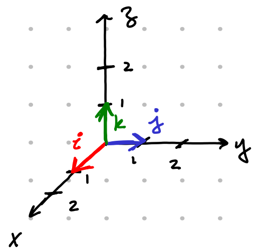
In n-dimensions, $\vec{v} = \langle x_1, x_2, \ldots, x_n\rangle$ $\quad |\vec{v}| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

Properties of Vectors

1. $\vec{a}+\vec{b} = \vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0} = \vec{a}$
4. $\vec{a}+(-\vec{a}) = \vec{0}$
5. $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a} = c\vec{a}+d\vec{b}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

"Vector space"

Standard basis vectors



$\{\vec{i}, \vec{j}, \vec{k}\}$

$\vec{i} : \langle 1, 0, 0\rangle$
$\vec{j} : \langle 0, 1, 0\rangle$
$\vec{k} : \langle 0, 0, 1\rangle$

$\vec{v} = A\cdot\vec{i} + B\cdot\vec{j} + C\cdot\vec{k}$

$\vec{v} = \langle 5, 7, 2\rangle = 5\vec{i} + 7\vec{j} + 2\vec{k}$

Examples + Practice Problems

r = radius $\quad (h,k,\ell)$ = center

Sphere: $r^2 = (x-h)^2 + (y-k)^2 + (z-\ell)^2$

Section 12.1

EXAMPLE 6 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

$$(x^2+4x+4) + (y^2-6y+9) + (z^2+2z+1) = -6+4+9+1$$

$$(x+2)^2 + (y-3)^2 + (z+1)^2 = 8$$

center $= (-2,3,-1)$ $\quad$ radius $= \sqrt{8} = 2\sqrt{2}$

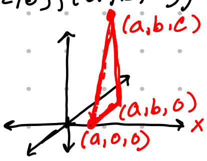
12. Find the distance from $(4, -2, 6)$ to each of the following.

(a) The xy-plane 6 $\quad$ (b) The yz-plane 4

(c) The xz-plane 2 $\quad$ (d) The x-axis 40

(e) The y-axis $\quad$ (f) The z-axis

$$d((4,-2,6),(a,b,0))^2 = (\cancel{4-a})^2 + (\cancel{-2-b})^2 + 6^2 \qquad a=4 \quad b=-2$$



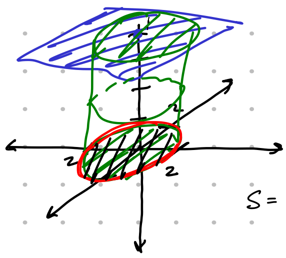
$P_1 = (4,-2,6)$

$P_2 = (4,0,0)$

$|P_1P_2| = \sqrt{0+4+36} = \sqrt{40}$

Write as inequalities:

40. The solid cylinder that lies on or below the plane $z = 8$ and on or above the disk in the xy-plane with center the origin and radius 2



$S = \{(x,y,z) \in \mathbb{R}^3 :$
$0 \le z \le 8$ & $x^2+y^2 \le 4\}$

41. The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$

$S = \{(x,y,z) \in \mathbb{R}^3 : r^2 < x^2+y^2+z^2 < R^2\}$

42. The solid upper hemisphere of the sphere of radius 2 centered at the origin

$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 4$ and $z > 0\}$

45. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

Hint: First think about what kind of shape this is.

$$\text{Midpoint}(AB) = \left(\frac{-1+6}{2}, \frac{5+2}{2}, \frac{3-2}{2}\right)$$
$$= \left(\frac{5}{2}, \frac{7}{2}, \frac{1}{2}\right)$$

Section 12.2

19–22 Find $\mathbf{a} + \mathbf{b}$, $4\mathbf{a} + 2\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

19. $\mathbf{a} = \langle -3, 4\rangle$, $\mathbf{b} = \langle 9, -1\rangle$ $\quad \vec{a}+\vec{b} = \langle -3+9, 4-1\rangle = \langle 6, 3\rangle$

20. $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -\mathbf{i} - 2\mathbf{j}$ $\quad \vec{a}+\vec{b} = \langle 5-1, 3-2\rangle = \langle 4, 1\rangle = 4\vec{i} + 1\vec{j}$

- $4\vec{a} = 4\langle -3, 4\rangle$
$= \langle -12, 16\rangle$

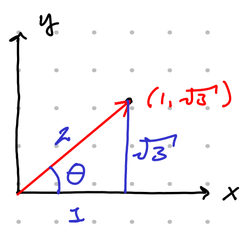
- Find a unit vector in the direction of $\langle 6,-2\rangle = v$

$|v| = \sqrt{36+4} = \sqrt{40}$

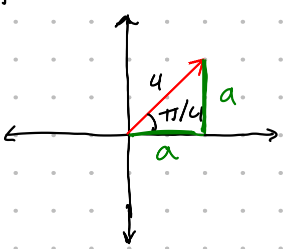
$\hat{v} = \frac{v}{|v|} = \frac{1}{\sqrt{40}}\langle 6,-2\rangle$

- What is the angle between $i + \sqrt{3}j$ and the positive x-axis?

$\tan^{-1}(\sqrt{3}) = \theta = 60° = \pi/3$



- If $\vec{v}$ lies in the first quadrant & makes an angle of $\pi/4$ with the positive x-axis, and $|\vec{v}| = 4$, write $\vec{v}$ in component form.



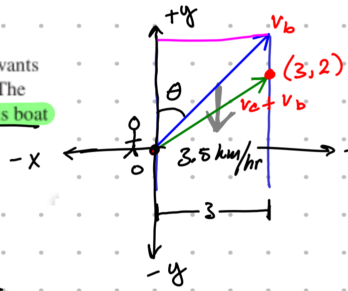
$\vec{v} = \_\_\,\vec{i} + \_\_\,\vec{j}$

$2a^2 = 4^2 \Rightarrow a^2 = 8 \Rightarrow a = 2\sqrt{2}$

$$4\cos(\pi/4)\vec{i} + 4\sin(\pi/4)\vec{j} = 2\sqrt{2}\,\vec{i} + 2\sqrt{2}\,\vec{j}$$

$\cos(\pi/4) = \frac{\sqrt{2}}{2}$ $\quad \sin(\pi/4) = \frac{\sqrt{2}}{2}$

39. A boatman wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from his starting point. The current in the canal flows at 3.5 km/h and the speed of his boat is 13 km/h.
(a) In what direction should he steer?
(b) How long will the trip take?



Speed of boat: 13km/hr

Step 1: Choose coordinates + label.

Step 2: Write things in vector notation

- velocity (boat) = $\langle a\sin\theta, b\cos\theta\rangle = v_b$ $\quad |v_b| = 13 \quad |v_b| = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = 13$

* What are a + b? $a = b = 13$ $\quad \vec{v}_b = \langle 13\sin\theta, 13\cos\theta\rangle$

- velocity (current) = $\langle 0, -3.5\rangle$ $\quad \vec{v}_c = \langle 0, -3.5\rangle$

* What is the "overall velocity"? $V = \vec{v}_c + \vec{v}_b = \langle 13\sin\theta, 13\cos\theta - 3.5\rangle$

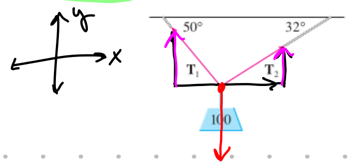
Step 3: Convert velocity to position via $\langle x, y\rangle = v \cdot t$
This will give you a system

Step 4: Solve for theta (43.4°) + time (.336 hours)

$$\langle 3, 2\rangle = \langle 13\sin\theta, 13\cos\theta - 3.5\rangle \cdot t = \langle 13\sin\theta \cdot t, 13\cos\theta \cdot t - 3.5t\rangle$$

$$\begin{cases} 3 = 13\sin\theta \cdot t \\ 2 = 13\cos\theta \cdot t - 3.5t \end{cases}$$

EXAMPLE 7 A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.



Step 1: Note that "weight" is just a measure of downward force. thus, a 100 lb - weight exerts a downward force $\vec{F}_{weight} = \langle 0, -100\rangle$ lbs

Step 2: Write $\vec{T}_1 + \vec{T}_2$ as vectors. Their sum = $\vec{F}_{weight}$

Step 3: Solve

$$\vec{T}_1 = -|\vec{T}_1|\cos(50°)\vec{i} + |\vec{T}_1|\sin(50°)\vec{j}$$

$$\vec{T}_2 = |\vec{T}_2|\cos(32°)\vec{i} + |\vec{T}_2|\sin(32°)\vec{j}$$

$$\vec{T}_1 + \vec{T}_2 = \vec{F}_{weight} = \langle 0, -100\rangle$$

$$|\vec{T}_1|\sin(50°) + |\vec{T}_2|\sin(32°) = 100$$

$$-|\vec{T}_1|\cos(50°) + |\vec{T}_2|\cos(32°) = 0$$

$$\boxed{\begin{aligned} T_1 &\approx -55.05\vec{i} + 65.60\vec{j} \\ T_2 &\approx 55.05\vec{i} + 34.40\vec{j} \end{aligned}}$$