

**MATH 201: LINEAR ALGEBRA**  
**SUGGESTED PROBLEMS FOR WEEK 9**

1. BASIC SKILLS

You will be expected to know all the definitions mentioned in this section *even if they were not covered in lecture*.

**Problem 1.1.** A *linear space* (or *vector space*) is a set endowed with an *addition operation* (+) and a *scalar multiplication operation* ( $\cdot$ ) which satisfy the following 8 conditions.

- (1)
- (2)
- (3)
- (4)
- (5)
- (6)
- (7)
- (8)

**Problem 1.2.** Give an example of...

- an infinite-dimensional linear space
- a 3-dimensional linear space other than  $\mathbb{R}^3$ .

**Problem 1.3.** Write definitions for the following terms.

- *neutral element* or *additive identity*
- *additive inverse*
- *addition operation*
- *scalar multiplication operation*
- *basis for a linear space*
- *subspace*
- *dimension of a linear space/subspace*

**Notation:** Let  $\{x_i\}_{i=1}^{\infty}$  denote an infinite sequence of numbers. For example, the sequence  $\{0, 2, 4, 6, \dots\}$  of even numbers can be written  $\{x_i\}_{i=1}^{\infty}$  where  $x_i = 2i - 2$ .

**Problem 1.4.** Let  $V$  be the set of infinite sequences of real numbers. Define

$$\{x_i\}_{i=1}^{\infty} + \{y_i\}_{i=1}^{\infty} = \{x_i + y_i\}_{i=1}^{\infty} \quad \text{and} \quad k\{x_i\}_{i=1}^{\infty} = \{kx_i\}_{i=1}^{\infty}.$$

Show that  $V$  is a linear space.

**Notation:** Let  $\mathbb{R}^{n \times m}$  denote the linear space of  $n \times m$  matrices where addition and scalar multiplication are defined as usual.

**Problem 1.5.** Write a basis for  $\mathbb{R}^{2 \times 3}$ .

**Problem 1.6.** A function  $T : V \rightarrow W$  between the linear spaces  $V$  and  $W$  is called a *linear transformation* if...

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**Problem 1.7.** Write a nontrivial example of...

- (a) a linear transformation  $T : C^{\infty} \rightarrow C^{\infty}$ .
- (b) a function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  that is *not* a linear transformation.
- (c) a linear transformation  $T : V \rightarrow V$  where  $V$  is as in Problem 1.4.

**Problem 1.8.** List *all* solutions to the differential equation

$$f''(x) + 2f'(x) + f(x) = 0.$$

Hint: Try solutions of the form  $e^{kx}$  and  $xe^{kx}$ . Then use Theorem 4.1.7 from Bretscher 4e.

## 2. TYPICAL PROBLEMS

**Notation:** Throughout this section,  $P_n$  will denote the set of polynomials of degree *at most*  $n$ . Similarly,  $P$  will denote the set of *all polynomials*.

**Problem 2.1.** Which of the following subsets of  $P_2$  are subspaces? Find a basis for those that are.

- (a)  $\{p(t) \mid \text{the degree of } p \text{ is } 2\}$
- (b)  $\{p(t) \mid p(0) = 2\}$
- (c)  $\{p(t) \mid p(2) = 0\}$
- (d)  $\left\{p(t) \mid \int_0^1 p(t) dt = 0\right\}$
- (e)  $\{p(t) \mid p(-t) = -p(t) \forall t\}$ .

**Problem 2.2.** Let  $V$  be the linear space of all infinite sequences of real numbers  $\{x_i\}_{i=1}^\infty$  (see problem 1.4). Which of the following subsets of  $V$  are subspaces? If possible, write a basis for each subspace and determine its dimension.

- (a) Arithmetic sequences: That is sequences of the form  $\{a, a + k, a + 2k, \dots\}$  for some constants  $a$  and  $k$ .
- (b) Geometric sequences: That is, sequences of the form  $\{a, ar, ar^2, ar^3, \dots\}$  for some constants  $a$  and  $r$ .
- (c) Square-summable sequences: That is, sequences  $\{x_0, x_1, \dots\}$  such that  $\sum_{i=0}^\infty x_i^2$  converges.

**Problem 2.3.** Suppose that  $B$  is an  $n \times n$  matrix with rank  $r$ . What is the dimension (in terms of  $n, r$ ) of the space of all  $n \times n$  matrices  $A$  such that  $BA = 0$ ?

**Problem 2.4.** If a matrix  $A$  represents the reflection across a line  $L$  (which passes through the origin) in  $\mathbb{R}^2$ , what is the dimension of the space of all matrices  $S$  such that

$$AS = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}?$$

**Problem 2.5.** Find a basis of each of the following linear spaces and thus determine their dimensions.

- (a)  $\{f \in P_4 \mid f(-x) = f(x)\}$
- (b)  $\{f \in P_4 \mid f(-x) = -f(x)\}$
- (c)  $\{f \in P_n \mid f(-x) = f(x)\}$
- (d)  $\{f \in P_n \mid f(-x) = -f(x)\}$

**Problem 2.6.** An *invertible linear transformation*  $T : V \rightarrow W$  is called an *isomorphism*. Consider the following functions. Determine whether they are linear transformations. If yes, determine if they are isomorphisms and calculate their image and kernel.

(a)

$$\begin{aligned} T : \mathbb{R}^{2 \times 2} &\rightarrow \mathbb{R} \\ M &\mapsto \det M. \end{aligned}$$

(b)

$$\begin{aligned} T : P_2 &\rightarrow P_2 \\ p(x) &\mapsto p'(x) + x^2. \end{aligned}$$

(c)

$$\begin{aligned} T : C^\infty &\rightarrow C^\infty \\ f &\mapsto f' + f''. \end{aligned}$$

(d)

$$\begin{aligned} T : P &\rightarrow P \\ p(x) &\mapsto \int_0^x p(t) dt \end{aligned}$$

(e)

$$\begin{aligned} T : P_2 &\rightarrow P_2 \\ p(x) &\mapsto p(-x). \end{aligned}$$

**Note:** For the next problem, you may use the following theorem without proof. The proof is worth studying!

**Theorem.** Let  $V$  and  $W$  be finite-dimensional linear spaces over  $\mathbb{R}$ . Then  $\dim V = \dim W = n$  if and only if there exists an isomorphism  $T : V \rightarrow W$ .

*Proof.* ( $\Rightarrow$ ) Assume that  $V$  and  $W$  have the same dimension. Choose a basis  $\{v_1, \dots, v_n\}$  of  $V$  and a basis  $\{w_1, \dots, w_n\}$  of  $W$ . Define  $T : V \rightarrow W$  by

$$T\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^n a_i w_i \quad (a_i \in \mathbb{R}).$$

This is well-defined because each vector of  $V$  has a unique expression in the basis  $\{v_i\}$ . Linearity follows from the definition.

- Injectivity: if  $T(\sum a_i v_i) = 0$ , then  $\sum a_i w_i = 0$ . Since  $\{w_i\}$  is a basis, all  $a_i = 0$ , so the only vector mapped to 0 is 0. Thus  $\ker T = \{0\}$ .
- Surjectivity: given  $w \in W$ , write  $w = \sum b_i w_i$ . Then  $v := \sum b_i v_i \in V$  satisfies  $T(v) = w$ . Hence  $\text{im } T = W$ .

Therefore  $T$  is linear and bijective, so it is an isomorphism.

( $\Leftarrow$ ) Assume there is an isomorphism  $T : V \rightarrow W$ , then  $\ker T = \{0\}$  and  $\text{im } T = W$ . By rank-nullity,

$$\dim V = \dim(\ker T) + \dim(\text{im } T) = 0 + \dim W,$$

so  $\dim V = \dim W$ .

Thus two finite-dimensional linear spaces are isomorphic if and only if they have the same dimension as desired.  $\square$

**Problem 2.7.** Define an isomorphism between  $P_3$  and  $\mathbb{R}^{2 \times 2}$  or show that it is not possible.

**Problem 2.8.** For which constants  $k$  is the linear transformation

$$T(M) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} M - M \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix}$$

an isomorphism from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$ ?

**Problem 2.9.** For which real numbers  $c_0, c_1, \dots, c_n$  is the linear transformation

$$T(f(t)) = \begin{bmatrix} f(c_0) \\ f(c_1) \\ \vdots \\ f(c_n) \end{bmatrix}$$

an isomorphism between  $P_n$  and  $\mathbb{R}^{n+1}$ ?

**Problem 2.10.** Let  $\mathbb{R}^+$  be the set of positive real numbers. On  $\mathbb{R}^+$  we define the operations

$$x \oplus y = xy$$

and

$$k \odot x = x^k.$$

- Show that  $\mathbb{R}^+$ , equipped with these operations, is a linear space. Find a basis for this space.
- Show that  $T(x) = \ln(x)$  is a linear transformation from  $\mathbb{R}^+$  to  $\mathbb{R}$ , where  $\mathbb{R}$  is endowed with the ordinary operations. Is  $T$  an isomorphism?

## 3. COMPLEX NUMBERS

In this section, we discuss an important linear space whose elements are referred to as *complex numbers*. Over the course of the following ten exercises, you will learn what a complex number is and some of its properties.

**Problem 3.1.** Let  $\mathbb{C} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ . Show that  $\mathbb{C}$  is a *subspace* of  $\mathbb{R}^{2 \times 2}$ .

**Problem 3.2.** For  $a, b \in \mathbb{R}$ , write

$$\langle a, b \rangle = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Compute

- $\langle a, b \rangle + \langle c, d \rangle$
- $\langle a, b \rangle \cdot \langle c, d \rangle$ .

Express your answers again in the form  $\langle \cdot, \cdot \rangle$ .

**Problem 3.3.** Define the *magnitude* of  $\langle a, b \rangle$  by

$$|\langle a, b \rangle| = \sqrt{\det(\langle a, b \rangle)}.$$

- (a) Show that  $|\langle a, b \rangle| = \sqrt{a^2 + b^2}$ .  
 (b) Show that  $|\langle a, b \rangle \cdot \langle c, d \rangle| = |\langle a, b \rangle| \cdot |\langle c, d \rangle|$ .

**Problem 3.4.** Define

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that  $\{I, J\}$  is a *basis* for  $\mathbb{C}$ .

- (a) Compute  $J^2$  in terms of  $I$ .  
 (b) For  $\vec{v} \in \mathbb{R}^2$ , what is the relationship between  $\vec{v}$  and  $J\vec{v}$ ?

**Problem 3.5.** Let  $r = \sqrt{a^2 + b^2}$ . Let  $\theta$  satisfy  $\cos \theta = \frac{a}{r}$ ,  $\sin \theta = \frac{b}{r}$  when  $r > 0$ . Show

$$\langle a, b \rangle = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

**Problem 3.6.** Let  $\langle a, b \rangle$  and  $\langle c, d \rangle$  in  $\mathbb{C}$ . Let  $r = |\langle a, b \rangle|$  and  $s = |\langle c, d \rangle|$ . Using the previous problem, let  $\theta$  be the angle associated to  $\langle a, b \rangle$  and  $\varphi$  the angle associated to  $\langle c, d \rangle$ . Show that

- (a)  $|\langle a, b \rangle \cdot \langle c, d \rangle| = rs$   
 (b) The angle associated to  $\langle a, b \rangle \cdot \langle c, d \rangle$  is  $\theta + \varphi$ .

**Problem 3.7.** Define the *conjugate* of  $\langle a, b \rangle$  to be  $\langle a, -b \rangle$ . Denote the conjugate of  $\langle a, b \rangle$  by  $\overline{\langle a, b \rangle}$ .

- (a) Show that  $\langle a, b \rangle \cdot \overline{\langle a, b \rangle} = (a^2 + b^2)I$ .  
 (b) In the case where  $\langle a, b \rangle \neq \langle 0, 0 \rangle$ , find a formula for  $\langle a, b \rangle^{-1}$ .

**Problem 3.8.** Solve for  $\langle x, y \rangle$  in the equation

$$\langle 3, 1 \rangle \cdot \langle x, y \rangle = \langle 2, -5 \rangle.$$

**Problem 3.9.** Let  $M = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$  be any real 2 by 2 matrix which satisfies

$$MJ = JM.$$

Prove that  $M \in \mathbb{C}$ .

**Problem 3.10.** The *subspace* spanned by the element  $J$  in  $\mathbb{C}$  is called the *imaginary numbers*. The *subspace* spanned by the element  $I$  in  $\mathbb{C}$  is called the *real numbers*. Since  $\{I, J\}$  form a basis for  $\mathbb{C}$ , any element  $z \in \mathbb{C}$  can be written in the form

$$z = aI + bJ.$$

Such an element is called a *complex number*. It is the convention to write it as  $z = a + bi$ . We refer to  $a$  as the *real part* of  $z$  and  $b$  as the *imaginary part* of  $z$ . Show that

$$z_1 \cdot z_2 = (a_1 + b_1i)(a_2 + b_2i) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2).$$

## 4. CHALLENGE PROBLEMS

**Problem 4.1.** Prove Theorem 4.1.7 from Bretscher 4e. That is, prove that the set of solutions to a differential equation of the form

$$f''(x) + af'(x) + bf(x) = 0$$

is a *two-dimensional linear space*.

**Problem 4.2.**

- (a) Show that there exists a *bijection* (that is, a one-to-one and onto map) from  $\mathbb{R}^n$  to  $\mathbb{R}$ .
- (b) Prove that it is possible to define “exotic” operations on  $\mathbb{R}^n$  such that  $\dim \mathbb{R}^n = 1$