MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 6

Problem 1. Determine whether the statements are true or false. Justify your answer.

(1) The column space of a matrix A is the set of solutions of $A\vec{x} = \vec{b}$.
(2) The system $A\vec{x} = \vec{b}$ is inconsistent if and only if $\vec{b}$ is *not* in the column space of A.
(3) The formula $(\ker B)^{\perp} = \operatorname{Im}\left(B^{\top}\right)$ holds for all matrices.
(4) Let A be an $m \times n$ matrix. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be the *row vectors* of A. Set $W = \operatorname{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$. Then

$$\vec{u} \cdot \vec{w} = 0 \qquad \forall\, \vec{u} \in \ker A \text{ and } \vec{w} \in W.$$

Problem 2. Show that the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.

Problem 3. Consider

$$\vec{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \qquad \mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \vec{c}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

(1) Find the *coordinate vectors* $\vec{x}_{\mathcal{B}}$ and $\vec{x}_{\mathcal{C}}$.
(2) Find a matrix A such that $A\,[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$ for *al* vectors $\vec{x}$.
(3) Find a matrix B such that $B\,[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$ for *all* vectors $\vec{x}$.

Problem 4. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -8 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Give a basis for

(1) $\operatorname{row}(A)$ (the span of the row vectors of A)
(2) $\operatorname{col}(A)$ (the span of the column vectors of A)
(3) $\ker(A)$.

Problem 5. Consider the matrix.

$$A = \begin{bmatrix} 1 & 2 & 6 & 0 & 1 \\ 1 & 2 & 9 & 1 & 2 \\ 1 & 2 & 12 & 2 & 3 \end{bmatrix}.$$

(1) Write *all independent* relations among the column vectors of A. Try to do this *without* putting A in reduced row echelon form.
(2) Write a *basis* for $\ker A$.

Challenge Problem. Consider vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ and a linear map $T : \mathbb{R}^n \to \mathbb{R}^k$. Define

$$W \;=\; \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} \;\cap\; \ker(T).$$

Assume

$$A = [\vec{v}_1 \;\cdots\; \vec{v}_m], \qquad \operatorname{rank}(A) = r.$$

and the restriction $T|_{\operatorname{span}\{\vec{v}_1,\ldots,\vec{v}_m\}}$ has rank s.

(1) Prove that $\dim W = r - s$.
(2) Give an explicit algorithm that takes as input the columns of A and the matrix of T, and outputs a basis of W, with a correctness proof.
(3) For $\vec{w} = \sum_{i=1}^{m} c_i \vec{v}_i$, define $C(\vec{w}) = |\{i : c_i \neq 0\}|$. Prove or disprove:

There exists a basis of W in which ***every*** *basis vector $\vec{w}$ satisfies $C(\vec{w}) \leqslant r - s + 1$.*