MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 6

Background: Orthogonal complements and a fact about transposes.

From now on, the term *column space* will refer to the span of the column vectos of a matrix. Similarly, *row space* will refer to the span of the row vectors of a matrix. We have seen that the column space of a matrix is equal to its image. It is therefore natural to ask: what is the meaning of the *row space* of a matrix? Part 3 of problem 1 below explores this topic. We will need the following standard definition (which will come in handy later on as well).

Definition 0.1. Suppose that $W \subseteq \mathbb{R}^n$ is a subspace. Its *orthogonal complement* $W^\perp$ is defined to be

$$\begin{aligned} W^\perp &= \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \perp \vec{w} \ \forall \, \vec{w} \in W\} \\ &= \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \ \forall \, \vec{w} \in W\}. \end{aligned}$$

And we will also need the following standard lemma.

Lemma 0.2. *Let A be an $m \times n$ matrix, $\vec{v} \in \mathbb{R}^n$, and $\vec{u} \in \mathbb{R}^m$. Then*

$$(A\vec{v}) \cdot \vec{u} = \vec{v} \cdot \left(A^\top \vec{u}\right).$$

Here, $A^\top$ is the transpose of the matrix A.

Proof. Note that for column vectors $\vec{x}$ and $\vec{y}$ with the same number of components,

$$\vec{x} \cdot \vec{y} = \vec{x}^\top \vec{y}.$$

On the left hand side of the above equation we have the dot product. On the right hand side, we have regular matrix multiplication. Thus

$$\begin{aligned} & (A\vec{v}) \cdot \vec{u} &=& \ (A\vec{v})^\top \vec{u} \\ \Rightarrow \quad & (A\vec{v}) \cdot \vec{u} &=& \ \vec{v}^\top A^\top \vec{u} = \vec{v} \cdot \left(A^\top \vec{u}\right). \end{aligned}$$

as desired. We have used the fact that $(AB)^\top = B^\top A^\top$. □

Problem 1. Determine whether the statements are true or false. Justify your answer.

(1) The column space of a matrix A is the set of solutions of $A\vec{x} = \vec{b}$.

Solution. False. The column space is the *image* or *outputs* of A while the set of solutions is the set of possile *inputs* for the equation $A\vec{x} = \vec{b}$.

(2) The system $A\vec{x} = \vec{b}$ is inconsistent if and only if $\vec{b}$ is *not* in the column space of A.

Solution. True. This follows from the fact that the image of a matrix A is equal to the span of its column vectors.

(3) The formula $(\ker B)^\perp = \operatorname{Im}\left(B^\top\right)$ holds for all matrices.

Solution. True. Here is a a proof. First, we prove that $(\ker B)^\perp \subseteq \operatorname{Im}\left(B^\top\right)$. Note that for *any* vector $\vec{x}$ and *any* $\vec{k} \in \ker B$, we have

$$\vec{x} \cdot B\vec{k} = 0$$

So, by lemma 0.2, we have

$$B^\top \vec{x} \cdot \vec{k} = 0.$$

Thus any vector $\vec{y}$ in the image of $B^\top$ is orthogonal to any vector in the kernel of B. The desired inclusion follows. Next, to conclude that $(\ker B)^\perp$ and $\operatorname{Im} B^\top$ are in fact the same sets, we prove that the have the same dimension. Note that for any subspace $V \subseteq \mathbb{R}^n$,

$$\dim V + \dim V^\perp = n.$$

Thus

$$\dim(\ker B)^{\perp} = n - \dim(\ker B).$$

The *rank-nulity theorem* tells us that

$$\dim(\ker B) + \operatorname{rank}(B) = n.$$

Therefore

$$\dim(\ker B)^{\perp} = \operatorname{rank}(B).$$

On the ohter hand, $\dim\left(\operatorname{Im}\left(B^{\top}\right)\right) = \operatorname{rank}\left(B^{\top}\right) = \operatorname{rank} B$.

Alternative, clearner argument: For any $\vec{x} \in \mathbb{R}^n$,

$$\begin{aligned} \vec{x} \in \operatorname{Im}\left(B^{\top}\right) &\Leftrightarrow \vec{x} \cdot B^{\top}\vec{y} = 0 \ \forall\, \vec{y} \in \mathbb{R}^n \\ &\Leftrightarrow B\vec{x} \cdot \vec{y} = 0 \ \forall\, \vec{y} \in \mathbb{R}^n \\ &\Leftrightarrow B\vec{x} = 0 \\ &\Leftrightarrow \vec{x} \in \ker B. \end{aligned}$$

Thus

$$\left(\operatorname{Im} A^{\top}\right)^{\perp} = \ker A.$$

Take the orthogonal complement of both sides, using the fact that $\left(W^{\perp}\right)^{\perp} = W$, to conclude that

$$(\ker B)^{\perp} = \operatorname{Im}\left(A^{\top}\right).$$

(4) Let A be an $m \times n$ matrix. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be the *row vectors* of A. Set $W = \operatorname{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$. Then

$$\vec{u} \cdot \vec{w} = 0 \qquad \forall\, \vec{u} \in \ker A \text{ and } \vec{w} \in W.$$

Solution. True. This just a re-wording of the previous part.

Problem 2. Show that the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.

Solution. We must show that these vectors are linearly independent. It suffices to show that the matrix whose rows are $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is invertible. We have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & -5 \end{bmatrix}.$$

This matrix is upper triangular and all the diagonal entries are nonzero. Thus the rank of the original matrix is 3. So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Problem 3. Consider

$$\vec{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \qquad \mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \vec{c}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

(1) Find the *coordinate vectors* $\vec{x}_{\mathcal{B}}$ and $\vec{x}_{\mathcal{C}}$.
(2) Find a matrix A such that $A[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$ for *al* vectors $\vec{x}$.
(3) Find a matrix B such that $B[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$ for *all* vectors $\vec{x}$.

Solution.

(1) We have $\vec{x} = 6\vec{b}_1 - 2\vec{b}_2 = 4\vec{c}_1 + 2\vec{c}_2$. Thus

$$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \qquad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

(2) In order to find the matrix A, we calculate $[\vec{e}_1]_{\mathcal{B}}, [\vec{e}_1]_{\mathcal{C}}, [\vec{e}_2]_{\mathcal{B}}, [\vec{e}_2]_{\mathcal{C}}$ where $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ are the standard basis vectors.

$$[\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix}1\\0\end{bmatrix} \qquad [\vec{e}_1]_{\mathcal{C}} = \begin{bmatrix}1\\1\end{bmatrix} \qquad [\vec{e}_2]_{\mathcal{B}} = \begin{bmatrix}-1\\1\end{bmatrix}, \qquad [\vec{e}_2]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}.$$

Next, we solve for the matrix $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ by calculating as follows.

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \Rightarrow a = 1, c = 1.$$

and

$$\begin{bmatrix}1 & b\\1 & d\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \Rightarrow b = 1, d = 2.$$

So

$$A = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}.$$

(3) The matrix B will be the *inverse* of the matrix A. Thus

$$B = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}.$$

Problem 4. Consider the matrix $A = \begin{bmatrix}1 & 1 & -8\\0 & 2 & 1\\1 & -1 & 0\end{bmatrix}$. Give a basis for

(1) row(A) (the span of the row vectors of A)
(2) col(A) (the span of the column vectors of A)
(3) ker(A).

Solution. Note that the matrix A is invertible since

$$\begin{bmatrix}1 & 1 & -8\\0 & 2 & 1\\1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 1 & -8\\0 & 2 & 1\\0 & -2 & 8\end{bmatrix} \to \begin{bmatrix}1 & 1 & -8\\0 & 2 & 1\\0 & 0 & 9\end{bmatrix}$$

upper triangular with nonzero diagonal entries. We thus conclude that the column space of A and the row space of A are both $\mathbb{R}^3$. So *any* set of three linearly independent vectors in $\mathbb{R}^3$ forms a basis for both. On the other hand, $\ker A = \left\{\vec{0}\right\}$ so a basis for the kernel of A is the *empty set*.

Problem 5. Consider the matrix.

$$A = \begin{bmatrix}1 & 2 & 6 & 0 & 1\\1 & 2 & 9 & 1 & 2\\1 & 2 & 12 & 2 & 3\end{bmatrix}.$$

(1) Write *all independent* relations among the column vectors of A. Try to do this *without* putting A in reduced row echelon form.
(2) Write a *basis* for $\ker A$.

Solution.

(1) There are three independent relations. They are
- $2\vec{v}_1 - \vec{v}_2 = \vec{0}$
- $3\vec{v}_2 - \vec{v}_3 + 3\vec{v}_4 = \vec{0}$
- $-\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = \vec{0}$.

Since there are at least two linearly independent column vectors, this is the complete list of nontrivial relations.

(2) The coefficients of the relations give rise to a basis for the kernel of A. That is,

$$\ker A = \operatorname{span}\left\{\begin{bmatrix}2\\-1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\3\\-1\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\-1\\1\end{bmatrix}\right\}.$$

Challenge Problem. Consider vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ and a linear map $T : \mathbb{R}^n \to \mathbb{R}^k$. Define

$$W = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} \cap \ker(T).$$

Assume

$$A = [\vec{v}_1 \ \cdots \ \vec{v}_m], \qquad \operatorname{rank}(A) = r.$$

and the restriction $T|_{\operatorname{span}\{\vec{v}_1,\ldots,\vec{v}_m\}}$ has rank s.

(1) Prove that $\dim W = r - s$.

(2) Give an explicit algorithm that takes as input the columns of A and the matrix of T, and outputs a basis of W, with a correctness proof.

(3) For $\vec{w} = \sum_{i=1}^m c_i\vec{v}_i$, define $C(\vec{w}) = |\{i : c_i \neq 0\}|$. Prove or disprove:

*There exists a basis of W in which **every** basis vector $\vec{w}$ satisfies $C(\vec{w}) \leqslant r - s + 1$.*

Solution.

(1) Dimension formula. Let $S = \operatorname{span}\{v_1, \ldots, v_m\} \subseteq V$. Then $\dim S = r$ because $\operatorname{rank}(A) = r$. Consider the linear map

$$\phi = T|_S : S \to \mathbb{R}^k.$$

By definition,

$$\ker(\phi) = \{x \in S : T(x) = 0\} = S \cap \ker(T) = W.$$

We are told $\operatorname{rank}(\phi) = s$. By rank–nullity on the map ϕ,

$$\dim S = \operatorname{rank}(\phi) + \dim\ker(\phi).$$

Substituting $\dim S = r$, $\operatorname{rank}(\phi) = s$, and $\ker(\phi) = W$ gives

$$r = s + \dim W,$$

hence

$$\dim W = r - s.$$

(2) Constructing a basis of W from A and T.

Let A be the $n \times m$ matrix with columns v_i. Any vector in $S = \operatorname{im}(A)$ can be written as $x = Ac$ for some $c \in \mathbb{R}^m$. The condition $x \in \ker(T)$ is

$$T(Ac) = 0 \iff (TA)c = 0.$$

So coefficients c that produce vectors in W are precisely

$$\mathcal{N} = \ker(TA) \subseteq \mathbb{R}^m,$$

and

$$W = \{Ac : c \in \mathcal{N}\} = A(\mathcal{N}).$$

Algorithm.

(1) Compute the matrix product $M = TA$ (a $k \times m$ matrix).

(2) Find a basis $\{c^{(1)}, \ldots, c^{(q)}\}$ of $\ker(M)$ by row-reducing M and parametrizing solutions of $Mc = 0$.

(3) Output the vectors

$$w_j = Ac^{(j)} \in \mathbb{R}^n, \qquad j = 1, \ldots, q.$$

(4) From $\{w_1, \ldots, w_q\}$, extract a maximal linearly independent subset (e.g. put them as columns of a matrix and row-reduce) and output that subset as a basis of W.

Correctness. First, for each j,

$$w_j = Ac^{(j)} \in \operatorname{im}(A) = S,$$

so $w_j \in \operatorname{span}\{v_1, \dots, v_m\}$. Also,

$$T(w_j) = T(Ac^{(j)}) = (TA)c^{(j)} = Mc^{(j)} = 0,$$

so $w_j \in \ker(T)$. Hence $w_j \in W$.

Next, let $w \in W$. Since $w \in S = \operatorname{im}(A)$, there exists $c \in \mathbb{R}^m$ with $w = Ac$. Since $w \in \ker(T)$, we have $0 = T(w) = T(Ac) = (TA)c$, so $c \in \ker(TA) = \mathcal{N}$. Writing c in the basis of $\mathcal{N}$,

$$c = \sum_{j=1}^{q} \alpha_j c^{(j)},$$

we get

$$w = Ac = \sum_{j=1}^{q} \alpha_j Ac^{(j)} = \sum_{j=1}^{q} \alpha_j w_j.$$

Thus $\{w_1, \dots, w_q\}$ spans W. Extracting a maximal independent subset yields a basis of W.

Finally, the size of a basis produced equals $\dim W = r - s$ by part (1), so the extraction step will leave exactly $r - s$ vectors.

(3) We exhibit a counterexample Take $n = m$ and let $A = I_m$, so $v_i = e_i$ and $r = \operatorname{rank}(A) = m$. Let $T : \mathbb{R}^m \to \mathbb{R}^{m-1}$ be defined by the $(m-1) \times m$ matrix

$$T = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

Then $Tx = 0$ forces $x_1 = x_2 = \cdots = x_m$, so

$$\ker(T) = \operatorname{span}\{(1, 1, \dots, 1)\},$$

hence $\dim \ker(T) = 1$ and $\operatorname{rank}(T) = m - 1$. Since $S = \operatorname{span}\{e_1, \dots, e_m\} = \mathbb{R}^m$, the restriction $T|_S$ has rank $s = m - 1$. Therefore

$$r - s = m - (m - 1) = 1,$$

so $W = S \cap \ker(T) = \ker(T)$ is 1-dimensional.

Any basis of W consists of a single nonzero vector, necessarily a nonzero multiple of $(1, 1, \dots, 1)$. Writing it as a combination of the $v_i = e_i$,

$$(1, 1, \dots, 1) = \sum_{i=1}^{m} 1 \cdot e_i,$$

so its support is m. But the proposed bound would require a basis vector with support

$$\leqslant r - s + 1 = 2.$$

For $m \geqslant 3$, this is impossible. Hence the statement is false.

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