MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 5

1. Basic Skills

Problem 1.1. Fill in the blanks.

(a) The *image* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\operatorname{Im}(T) = \{ \quad \underline{\hspace{3cm}} \quad | \quad \underline{\hspace{3cm}} \quad \}$$

(b) The *kernel* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\ker(T) = \{ \quad \underline{\hspace{3cm}} \quad | \quad \underline{\hspace{3cm}} \quad \}.$$

Note: If $T(\vec{x}) = A\vec{x}$ for some matrix A, we also write $\operatorname{Im}(A) = \operatorname{Im}(T)$ and $\ker(A) = \ker(T)$.

Problem 1.2. Suppose that $T_A(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(a) Write a set of vectors that *span* the image of T_A.
(b) Write a set of vectors that *span* the kernel of T_A.
(c) What is the *minimum* number of vectors needed to span the image of T_A?
(d) What is the *minimum* number of vectors needed to span the kernel of T_A?

Problem 1.3. Suppose that A is a square matrix. Write 8 separate statements equivalent to the statement

"A is invertible."

In other words, review "Summary 3.3.10" from Bretscher 4th edition.

Problem 1.4. Which of the following sets are *subspaces* of $\mathbb{R}^3$?

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 1 \right\}$.

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x \leqslant y \leqslant z \right\}$.

(c) $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} \mid x, y, z \text{ are arbitrary constants} \right\}$.

Problem 1.5. Consider the following lists of vectors. For each list, determine whether the given vectors are linear independent.

(a) $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$.

2. Typical Problems

Problem 2.1. Write a linear transformation whose kernel is the line spanned by the vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Problem 2.2. Write a linear transformation whose kernel is the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

Problem 2.3. Consider an $n \times p$ matrix A and a $p \times m$ matrix B such that $\ker A = \left\{\vec{0}\right\}$ and $\ker B = \left\{\vec{0}\right\}$. What is $\ker AB$?

Problem 2.4. Let A be a matrix and let $B = \text{rref}(A)$.

(a) Is $\ker A$ necessarily equal to $\ker B$? Explain.
(b) Is $\text{Im}A$ necessarily equal to $\text{Im}B$? Explain.

Problem 2.5. Consider two subspaces V and W of $\mathbb{R}^n$.

(a) Is $V \cup W$ a subspace of $\mathbb{R}^n$?
(b) Is $V \cap W$ a subspace of $\mathbb{R}^n$?

Problem 2.6. Let $T : \mathbb{R}^n \to \mathbb{R}^p$ be a linear transformation. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ are linearly independent. Under what conditions (on T, m, n, p) are $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ linearly independent?

Problem 2.7. Find a *basis* for the image of the matrices

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$.

Problem 2.8. Consider an $m \times n$ matrix A and an $n \times m$ matrix B with $n \neq m$ such that $AB = I_m$. Are the columns of B linearly independent? What about the columns of A?

Problem 2.9. Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$ are linearly independent. Are the vecotrs $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ linearly independent?

Problem 2.10. For which values of the constants a, b, c, d, e and f are the following vectors linearly independent?

$$\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} d \\ e \\ f \\ 0 \end{bmatrix}.$$

3. Proofs

Write a *rigorous, mathematical proof* for each of the claims below.

Problem 3.1. Suppose that A is an $n \times n$ matrix such that $A^2 = 0$. Then $\text{Im}(A) \subseteq \ker(A)$. Consequently, $\text{rank}(A) \leqslant \frac{n}{2}$

Problem 3.2. Suppose that $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations such that $S \circ T = 0$ and $S + T = I$. Then $\text{Im}(T) = \ker(S)$ and $\text{Im}(S) = \ker(T)$.

Problem 3.3. For any $m \times n$ matrix A and $n \times p$ matrix B,

$$\text{rank}(AB) \leqslant \min \{\text{rank}(A), \text{rank}(B)\}.$$

Problem 3.4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $T^2 = T$. Define $S = I - T$.

(a) Show that $S^2 = S$.
(b) Show that $\ker(T) = \text{Im}(S)$ and $\ker(S) = \text{Im}(T)$.

Problem 3.5. Suppose that A and B are $n \times n$ matrices. Suppose that (AB) is invertible. Show that A and B are invertible.

Problem 3.6. Suppose that A is an $n \times n$ matrix. Prove that $\text{Im}(A^2) \subseteq \text{Im}(A)$

Problem 3.7. True of false? There exists a 2×2 matrix A such that $A^2 \neq 0$ *and* $A^3 = 0$.

4. Challenge Problems

Note: The challenge problems this week are *very doable and everyone should try them!*

Problem 4.1. Suppose that you are given six vectors $v_1, \ldots, v_6 \in \mathbb{R}^5$ with the following properties.

- Any five of them span $\mathbb{R}^5$.
- The only *relation* among them is $v_1 + v_2 + \cdots + v_6 = 0$.

Let A be the 5×6 matrix whose column vectors are v_i.

(a) What is $\dim(\text{Im}(A))$ and $\dim(\ker(A))$?
(b) Pick *any* index j. Consider the 5×5 matrix A^j obtained by deleting column j from A. Is A^j invertible?
(c) Define $T : \mathbb{R}^6 \to \mathbb{R}^5$ by $T(e_i) = v_i$. For how many distinct scalars c does there exist a linear map $f : \mathbb{R}^5 \to \mathbb{R}$ with $f(v_i) = c$ for all i?

Problem 4.2. Let $n \geqslant 2$. Suppose that a linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $S^2 = 0$ and $\dim(\text{Im}(S)) = k$ for some $k \geqslant 1$. Define

$$T(\vec{x}) = \vec{x} + S(\vec{x}).$$

(a) Compute $\dim(\ker T)$ and $\dim(\text{Im} T)$.
(b) Describe a basis of $\mathbb{R}^n$ in which the matrix of T has the simplest possible block form. State that form.
(c) For which positive integers m does T^m have the *same* image as T? For which m does T^m have the *same* kernel as T?

Problem 4.3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Call T **decisive** if every nonzero vector in $\mathbb{R}^n$ belongs to exactly one of $\text{Im}(T)$ or $\ker(T)$. Determine all decisive linear transformations.