MATH 201: LINEAR ALGEBRA
SUGGESTED PROBLEMS FOR WEEK 5

1. Basic Skills

Problem 1.1. Fill in the blanks.

(a) The *image* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\mathrm{Im}(T) = \{ \quad \underline{\hspace{3cm}} \quad | \quad \underline{\hspace{3cm}} \quad \}$$

(b) The *kernel* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\ker(T) = \{ \quad \underline{\hspace{3cm}} \quad | \quad \underline{\hspace{3cm}} \quad \}.$$

Solution.
(a) $\mathrm{Im}(T) = \{ T(x) : x \in \mathbb{R}^m \}$, the set of all outputs of T.
(b) $\ker(T) = \{ x \in \mathbb{R}^m : T(x) = 0 \}$, the set of inputs sent to the zero vector.

Problem 1.2. Suppose that $T_A(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(a) Write a set of vectors that *span* the image of T_A.
(b) Write a set of vectors that *span* the kernel of T_A.
(c) What is the *minimum* number of vectors needed to span the image of T_A?
(d) What is the *minimum* number of vectors needed to span the kernel of T_A?

Solution.
Write $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. The columns are

$$c_1 = (1,1)^\top, \quad c_2 = (1,2)^\top, \quad c_3 = (1,3)^\top, \quad c_4 = (1,4)^\top$$

(a) $\mathrm{Im}(T_A) = \mathrm{span}\{c_1, c_2, c_3, c_4\} = \mathrm{span}\{(1,1)^\top, (1,2)^\top\}$, since any two noncollinear columns span $\mathbb{R}^2$.
(b) Solve $Ax = 0$ with $x = (x_1, x_2, x_3, x_4)^\top$. Row-reduce $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, so $x_1 = x_3 + 2x_4$, $x_2 = -2x_3 - 3x_4$. Let

$$s = x_3, \quad t = x_4.$$

Then

$$x = s(1,-2,1,0)^\top + t(2,-3,0,1)^\top.$$

Hence a spanning set for $\ker T_A$ is $\{(1,-2,1,0)^\top, (2,-3,0,1)^\top\}$.
(c) $\mathrm{rank}(A) = 2$, so the minimum number is 2.
(d) $\mathrm{nullity}(A) = 4 - 2 = 2$, so the minimum number is 2.

Problem 1.3. Suppose that A is a square matrix. Write 8 separate statements equivalent to the statement

"A is invertible."

In other words, review "Summary 3.3.10" from Bretscher 4th edition.

Solution.
Possible answers (any correct eight earn full credit):

(1) A has an inverse matrix A^{-1}.
(2) $\det A \neq 0$ (or equivalently, A row-reduces to I).
(3) The columns of A are linearly independent.
(4) The columns of A span $\mathbb{R}^n$.
(5) $\mathrm{rank}(A) = n$.

(6) $\ker A = \{0\}$.
(7) The only solution to $Ax = 0$ is $x = 0$.
(8) For every $b \in \mathbb{R}^n$, the system $Ax = b$ has a unique solution.
(9) A is a product of elementary matrices.
(10) The linear map $x \mapsto Ax$ is one-to-one and onto.

(Write any eight; keep your phrasing consistent with what you've learned in class.)

Problem 1.4. Which of the following sets are *subspaces* of $\mathbb{R}^3$?

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 1 \right\}$.

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x \leqslant y \leqslant z \right\}$.

(c) $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} \mid x, y, z \text{ are arbitrary constants} \right\}$.

Solution.

(a) Not a subspace: it does not contain 0 (since $0 + 0 + 0 \neq 1$) and is not closed under scaling.
(b) Not a subspace: not closed under scaling (multiply by -1 reverses the inequalities).
(c) Yes, it is a subspace: it is the span of the three column vectors $(1, 4, 7)^\top$, $(2, 5, 8)^\top$, $(3, 6, 9)^\top$.

Problem 1.5. Consider the following lists of vectors. For each list, determine whether the given vectors are linear independent.

(a) $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$.

Solution.
(a) Dependent (one vector is the zero vector).
(b) Dependent: the first two are multiples; the last vector is the sum of 3 times the first, 4 times the third, and 5 times the fourth.
(c) Try to solve:

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}.$$

We have

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

This indicates that $a = 3, b = 1$ is a solution. That is, labeling the vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, we have $\vec{v}_3 = 3\vec{v}_1 + \vec{v}_2$.

2. Typical Problems

Problem 2.1. Write a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is the line spanned by the vector $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Solution. Suppose that the 3×3 matrix A satisfies $A\vec{v} = 0$. Let $\vec{c}_1, \vec{c}_2, \vec{c}_3$ be the column vectors of A. Then

$$\begin{aligned} A\vec{v} &= -\vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 = 0 \\ &\Rightarrow \vec{c}_1 = \vec{c}_2 + 2\vec{c}_3. \end{aligned}$$

Thus

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is one example.

Problem 2.2. Write a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is the plane $x+2y+3z = 0$ in $\mathbb{R}^3$.

Solution. The image of T should be *normal* to the plane. The normal vector of this plane is $\langle 1, 2, 3\rangle$. To see this, note that $\langle x, y, z\rangle \cdot \langle 1, 2, 3\rangle = x + 2y + 3z = 0$ is the plane equation given. Next, recall that the image of a linear transformation is equal to the *span* of its column vectors. Thus it suffices to construct a matrix A whose image is equal to the span of $\langle 1, 2, 3\rangle$. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Problem 2.3. Consider an $n \times p$ matrix A and a $p \times m$ matrix B such that $\ker A = \left\{\vec{0}\right\}$ and $\ker B = \left\{\vec{0}\right\}$. What is $\ker AB$?

Solution.
If $\ker A = \{0\}$ and $\ker B = \{0\}$, then A is injective on its domain and B is injective on its domain. If $ABx = 0$, then $A(Bx) = 0$, hence $Bx = 0$ (injectivity of A), hence $x = 0$ (injectivity of B). Therefore $\ker(AB) = \{0\}$.

Problem 2.4. Let A be a matrix and let $B = \text{rref}(A)$.
(a) Is $\ker A$ necessarily equal to $\ker B$? Explain.
(b) Is $\text{Im}A$ necessarily equal to $\text{Im}B$? Explain.

Solution.
(a) $\ker A = \ker(\text{rref}(A))$. Row operations preserve the solution set of $Ax = 0$, so the nullspace is unchanged.
(b) In general $\text{Im}\, A \neq \text{Im}(\text{rref}(A))$. Row operations mix rows (i.e., change the span of the rows, not the columns). Counterexample: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has image $\text{span}\{(1, 1)^\top\}$, but $\text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has image $\text{span}\{(1, 0)^\top\}$.

Problem 2.5. Consider two subspaces V and W of $\mathbb{R}^n$.
(a) Is $V \cup W$ a subspace of $\mathbb{R}^n$?
(b) Is $V \cap W$ a subspace of $\mathbb{R}^n$?

Solution.
(a) Typically no: $V \cup W$ need not be closed under addition. Example in $\mathbb{R}^2$: $V = x$-axis, $W = y$-axis; $V \cup W$ is not a subspace.
(b) Yes: $V \cap W$ is always a subspace (it contains 0 and is closed under addition and scaling).

Problem 2.6. Let $T : \mathbb{R}^n \to \mathbb{R}^p$ be a linear transformation. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ are linearly independent. Under what conditions (on T, m, n, p) are $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ linearly independent?

Solution. Claim: $T(\vec{v}_i)$ are linearly independent if and only if

$$\ker T \cap \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = \left\{\vec{0}\right\}.$$

Proof: First, we prove that if $T(\vec{v}_i)$ are *not* linearly independent than $\ker T \cap \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$ is nonempty. Suppose that $\sum_i a_i T(\vec{v}_i) = 0$. Then $T\left(\sum_i a_i \vec{v}_i\right) = 0$ by linearity. Thus $\sum_i a_i \vec{v}_i \in \ker T \cap \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$. Next we prove the converse. That is, we prove that if $\ker T \cap \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = \left\{\vec{0}\right\}$ then $T(\vec{v}_i)$ are *not* linearly independent. Suppose there exists a nonzero $w \in \ker T \cap \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$. Then $w = \sum_i a_i \vec{v}_i$ and $T(w) = 0$ so we have a nontrivial relation among $T(\vec{v}_i)$ as desired.

□

Problem 2.7. Find a *basis* for the image of the matrices

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$.

Solution.
(a) Columns $\{(1,1,1)^\top, (1,2,3)^\top\}$ are linearly independent; they form a basis for the image.
(b) Row-reduce $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, so rank $= 2$. A basis for the image is $\{(1,1,1)^\top, (1,2,3)^\top\}$ (the first two pivot columns).
(c)Note that the matrix is in ref. There are *three* leading ones. We take the three pivot columns:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(d) In this case, there are two columns and the second is not a multiple of the first. Thus they are a basis for the image.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

Problem 2.8. Consider an $m \times n$ matrix A and an $n \times m$ matrix B with $n \neq m$ such that $AB = I_m$. Are the columns of B linearly independent? What about the columns of A?

Solution.
If $AB = I_m$, then B has linearly independent columns (since $ABx = 0 \Rightarrow x = 0$). Also A has linearly independent rows (equivalently, rank m). The columns of A need not be independent when $n \neq m$ (e.g., A is $m \times n$ with $m < n$ can have dependent columns).

Problem 2.9. Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$ are linearly independent. Are the vecotrs $\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ linearly independent?

Solution.

Yes, they are linearly independent if v_1, v_2, v_3 are independent. Suppose $av_1 + b(v_1 + v_2) + c(v_1 + v_2 + v_3) = 0$. Then $(a + b + c)v_1 + (b + c)v_2 + cv_3 = 0$. By independence, $c = 0$, then $b = 0$, then $a = 0$.

Problem 2.10. For which values of the constants a, b, c, d, e and f are the following vectors linearly independent?

$$\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} d \\ e \\ f \\ 0 \end{bmatrix}.$$

Solution.

Arrange the three vectors as columns of a 4×3 matrix $M = \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix}$. These columns are independent iff $\operatorname{rank}(M) = 3$, i.e., iff all diagonal entries a, c, f are nonzero. Thus the set is linearly independent exactly when $a \neq 0$, $c \neq 0$, and $f \neq 0$ (no conditions on b, d, e).

3. Proofs

Write a *rigorous, mathematical proof* for each of the claims below.

Problem 3.1. Suppose that A is an $n \times n$ matrix such that $A^2 = 0$. Then $\operatorname{Im}(A) \subseteq \ker(A)$. Consequently, $\operatorname{rank}(A) \leqslant \frac{n}{2}$

Solution. Let $\vec{y} \in \operatorname{Im}(A)$. Then $\vec{y} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. We compute:

$$A\vec{y} = A(A\vec{x}) = A^2\vec{x} = 0\vec{x} = \vec{0}.$$

Therefore $\vec{y} \in \ker(A)$. Since $\vec{y}$ was arbitrary, $\operatorname{Im}(A) \subseteq \ker(A)$. Now we derive the rank bound. Since $\operatorname{Im}(A) \subseteq \ker(A)$, we have

$$\operatorname{rank}(A) = \dim(\operatorname{Im}(A)) \leqslant \dim(\ker(A)) = \operatorname{nullity}(A).$$

By the rank-nullity theorem,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

Since $\operatorname{rank}(A) \leqslant \operatorname{nullity}(A)$, substituting gives

$$\operatorname{rank}(A) + \operatorname{nullity}(A) \geqslant \operatorname{rank}(A) + \operatorname{rank}(A) = 2\operatorname{rank}(A),$$

so $n \geqslant 2\operatorname{rank}(A)$, which yields $\operatorname{rank}(A) \leqslant \frac{n}{2}$. □

Problem 3.2. Suppose that $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations such that $S \circ T = 0$ and $S + T = I$. Then $\operatorname{Im}(T) = \ker(S)$ and $\operatorname{Im}(S) = \ker(T)$.

Solution. We prove $\operatorname{Im}(T) = \ker(S)$. The proof of $\operatorname{Im}(S) = \ker(T)$ is analogous (by symmetry, since $T \circ S = 0$ as well).

($\operatorname{Im}(T) \subseteq \ker(S)$): Let $\vec{y} \in \operatorname{Im}(T)$, so $\vec{y} = T(\vec{x})$ for some $\vec{x}$. Then

$$S(\vec{y}) = S(T(\vec{x})) = (S \circ T)(\vec{x}) = \vec{0}.$$

Hence $\vec{y} \in \ker(S)$.

($\ker(S) \subseteq \operatorname{Im}(T)$): Let $\vec{y} \in \ker(S)$, so $S(\vec{y}) = \vec{0}$. Since $S + T = I$, we have

$$\vec{y} = I(\vec{y}) = S(\vec{y}) + T(\vec{y}) = \vec{0} + T(\vec{y}) = T(\vec{y}).$$

Therefore $\vec{y} \in \operatorname{Im}(T)$.

Combining both inclusions, $\operatorname{Im}(T) = \ker(S)$.

Symmetry check: We verify $T \circ S = 0$. From $S + T = I$, we get $S = I - T$, so

$$T \circ S = T(I - T) = T - T^2.$$

Also, from $S + T = I$ and $S \circ T = 0$: applying $S + T = I$ to T on the right gives $ST + T^2 = T$, so $T^2 = T$ (since $ST = 0$). Thus $T \circ S = T - T^2 = 0$.

By the identical argument with the roles of S and T swapped, $\mathrm{Im}(S) = \ker(T)$. □

Problem 3.3. For any $m \times n$ matrix A and $n \times p$ matrix B,

$$\mathrm{rank}(AB) \leqslant \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

Solution. We establish each inequality separately.

Inequality 1: $\mathrm{rank}(AB) \leqslant \mathrm{rank}(A)$.

Let $\vec{y} \in \mathrm{Im}(AB)$. Then $\vec{y} = AB\vec{x}$ for some $\vec{x} \in \mathbb{R}^p$. Setting $\vec{z} = B\vec{x} \in \mathbb{R}^n$, we have $\vec{y} = A\vec{z} \in \mathrm{Im}(A)$. Therefore

$$\mathrm{Im}(AB) \subseteq \mathrm{Im}(A).$$

Since the dimension of a subspace contained in another cannot exceed the dimension of the larger one,

$$\mathrm{rank}(AB) = \dim(\mathrm{Im}(AB)) \leqslant \dim(\mathrm{Im}(A)) = \mathrm{rank}(A).$$

Inequality 2: $\mathrm{rank}(AB) \leqslant \mathrm{rank}(B)$.

Let $\vec{x} \in \ker(B)$, so $B\vec{x} = \vec{0}$. Then

$$AB\vec{x} = A\vec{0} = \vec{0},$$

so $\vec{x} \in \ker(AB)$. Therefore $\ker(B) \subseteq \ker(AB)$, which gives

$$\mathrm{nullity}(B) = \dim(\ker(B)) \leqslant \dim(\ker(AB)) = \mathrm{nullity}(AB).$$

By the rank-nullity theorem applied to B (a $n \times p$ matrix) and AB (a $m \times p$ matrix), both with domain $\mathbb{R}^p$:

$$\mathrm{rank}(B) + \mathrm{nullity}(B) = p = \mathrm{rank}(AB) + \mathrm{nullity}(AB).$$

Since $\mathrm{nullity}(B) \leqslant \mathrm{nullity}(AB)$, it follows that $\mathrm{rank}(AB) \leqslant \mathrm{rank}(B)$.

Combining both inequalities yields the result. □

Problem 3.4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $T^2 = T$. Define $S = I - T$.

(a) Show that $S^2 = S$.

(b) Show that $\ker(T) = \mathrm{Im}(S)$ and $\ker(S) = \mathrm{Im}(T)$.

Solution.

(a) We compute directly:

$$S^2 = (I - T)^2 = I - 2T + T^2 = I - 2T + T = I - T = S,$$

where we used $T^2 = T$ in the third equality. □

(b) We prove each of the two equalities by showing containment in both directions.

Proof that $\mathrm{Im}(S) \subseteq \ker(T)$: Let $\vec{y} \in \mathrm{Im}(S)$. Then $\vec{y} = S(\vec{x})$ for some $\vec{x} \in \mathbb{R}^n$. We compute

$$T(\vec{y}) = T(S(\vec{x})) = T(I - T)(\vec{x}) = (T - T^2)(\vec{x}) = (T - T)(\vec{x}) = \vec{0}.$$

Therefore $\vec{y} \in \ker(T)$.

Proof that $\ker(T) \subseteq \mathrm{Im}(S)$: Let $\vec{y} \in \ker(T)$, so $T(\vec{y}) = \vec{0}$. Since $S = I - T$, we have

$$S(\vec{y}) = \vec{y} - T(\vec{y}) = \vec{y} - \vec{0} = \vec{y}.$$

Therefore $\vec{y} = S(\vec{y}) \in \mathrm{Im}(S)$.

Combining both inclusions, $\ker(T) = \operatorname{Im}(S)$.

Proof that $\operatorname{Im}(T) \subseteq \ker(S)$: Let $\vec{y} \in \operatorname{Im}(T)$. Then $\vec{y} = T(\vec{x})$ for some $\vec{x} \in \mathbb{R}^n$. We compute

$$S(\vec{y}) = (I - T)(T(\vec{x})) = T(\vec{x}) - T^2(\vec{x}) = T(\vec{x}) - T(\vec{x}) = \vec{0}.$$

Therefore $\vec{y} \in \ker(S)$.

Proof that $\ker(S) \subseteq \operatorname{Im}(T)$: Let $\vec{y} \in \ker(S)$, so $S(\vec{y}) = \vec{0}$. Since $S = I - T$, this means

$$\vec{y} - T(\vec{y}) = \vec{0}, \quad \text{i.e.,} \quad \vec{y} = T(\vec{y}).$$

Therefore $\vec{y} = T(\vec{y}) \in \operatorname{Im}(T)$.

Combining both inclusions, $\ker(S) = \operatorname{Im}(T)$. □

Problem 3.5. Suppose that A and B are $n \times n$ matrices. Suppose that (AB) is invertible. Show that A and B are invertible.

Solution. We show each matrix is invertible by proving its kernel is trivial, then applying the rank-nullity theorem.

B is invertible: Suppose $B\vec{x} = \vec{0}$. Then

$$(AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}.$$

Since AB is invertible, $\ker(AB) = \{\vec{0}\}$, so $\vec{x} = \vec{0}$. Therefore $\ker(B) = \{\vec{0}\}$.

By the rank-nullity theorem, $\operatorname{rank}(B) + \operatorname{nullity}(B) = n$. Since $\operatorname{nullity}(B) = 0$, we get $\operatorname{rank}(B) = n$, so B is invertible.

A is invertible: Since AB is invertible, $(AB)^{-1}$ exists, and so

$$A \cdot \left(B \cdot (AB)^{-1}\right) = (AB)(AB)^{-1} = I.$$

Thus A has a right inverse $C = B(AB)^{-1}$. We now show $\ker(A) = \{\vec{0}\}$.

Suppose $A\vec{x} = \vec{0}$. Multiplying on the left by C^{T} is not useful, so instead we argue via dimension. Since $AC = I$, for any $\vec{y} \in \mathbb{R}^n$ we have $A(C\vec{y}) = \vec{y}$, so $\operatorname{Im}(A) = \mathbb{R}^n$. That is, $\operatorname{rank}(A) = n$. By the rank-nullity theorem, $\operatorname{nullity}(A) = 0$, so A is invertible.

Alternatively, here is a self-contained argument that avoids constructing a right inverse. Since AB is invertible, $\operatorname{Im}(AB) = \mathbb{R}^n$. For every $\vec{y} \in \mathbb{R}^n$ there exists $\vec{x}$ with $(AB)\vec{x} = \vec{y}$, i.e., $A(B\vec{x}) = \vec{y}$. Therefore every $\vec{y}$ is in $\operatorname{Im}(A)$, so $\operatorname{Im}(A) = \mathbb{R}^n$, giving $\operatorname{rank}(A) = n$. By rank-nullity, $\operatorname{nullity}(A) = 0$, so A is invertible. □

Problem 3.6. Suppose that A is an $n \times n$ matrix. Prove that $\operatorname{Im}(A^2) \subseteq \operatorname{Im}(A)$

Solution. Let $\vec{y} \in \operatorname{Im}(A^2)$. By definition, there exists $\vec{x} \in \mathbb{R}^n$ such that

$$\vec{y} = A^2\vec{x} = A(A\vec{x}).$$

Setting $\vec{z} = A\vec{x} \in \mathbb{R}^n$, we have $\vec{y} = A\vec{z}$, which shows $\vec{y} \in \operatorname{Im}(A)$.

Since $\vec{y}$ was an arbitrary element of $\operatorname{Im}(A^2)$, we conclude $\operatorname{Im}(A^2) \subseteq \operatorname{Im}(A)$. □

Remark. The same argument generalizes immediately: for any positive integers $j > k$, $\operatorname{Im}(A^j) \subseteq \operatorname{Im}(A^k)$, since $A^j\vec{x} = A^k(A^{j-k}\vec{x})$.

Problem 3.7. True of false? There exists a 2×2 matrix A such that $A^2 \neq 0$ *and* $A^3 = 0$.

Solution. False. Assume true. Let A be such a matrix. Then

- $\operatorname{rank} A = 1$ (since $\det(A^3) = (\det A)^3 = 0 \Rightarrow \det A = 0$.
- $\dim \ker A = 1$ (by rank nullity)
- $\operatorname{Im} A^2 \subseteq \operatorname{Im} A$ (fact from class)
- $\operatorname{Im} A^2 \subseteq \ker A$ since $A^3\vec{x} \quad A(A^2\vec{x}) = 0$
- $\operatorname{Im} A = \ker A$ (the two items above show that $\operatorname{Im} A \cap \ker A \neq \left\{\vec{0}\right\}$, and if two one dimensional subspaces have nontrivial intersection, they must me equal.)
- $A\left(A\vec{x}\right) = \vec{0}\ \forall \vec{x}$ since $A\vec{x} \in \operatorname{Im} A = \ker A$.

- contradiction.

□

4. Challenge Problems

Problem 4.1. Suppose that you are given six vectors $v_1, \dots, v_6 \in \mathbb{R}^5$ with the following properties.

- Any five of them span $\mathbb{R}^5$.
- The only *relation* among them is $v_1 + v_2 + \cdots + v_6 = 0$.

Let A be the 5×6 matrix whose column vectors are v_i.

(a) What is $\dim(\mathrm{Im}(A))$ and $\dim(\ker(A))$?
(b) Pick *any* index j. Consider the 5×5 matrix A^j obtained by deleting column j from A. Is A^j invertible?
(c) Define $T : \mathbb{R}^6 \to \mathbb{R}^5$ by $T(e_i) = v_i$. For how many distinct scalars c does there exist a linear map $f : \mathbb{R}^5 \to \mathbb{R}$ with $f(v_i) = c$ for all i?

Solution.
(a) Any five span $\mathbb{R}^5 \Rightarrow$ the image has dimension 5. By Rank–Nullity on $T : \mathbb{R}^6 \to \mathbb{R}^5$, $\dim \ker T = 6 - 5 = 1$.
(b) Deleting any one column removes the sole relation $v_1 + \cdots + v_6 = 0$, so the remaining 5 columns are independent in $\mathbb{R}^5$; hence each A_j is invertible.
(c) The condition "$f(v_i) = c$ for all i" means f is constant on the image of T. Since $\mathrm{Im}\, T = \mathbb{R}^5$, we must have $f(x) \equiv c$ as a linear map, which forces $c = 0$. Therefore there is exactly one scalar c (namely 0) for which such an f exists.

Problem 4.2. Let $n \geqslant 2$. Suppose that a linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $S^2 = 0$ and $\dim(\mathrm{Im}(S)) = k$ for some $k \geqslant 1$. Define

$$T(\vec{x}) = \vec{x} + S(\vec{x}).$$

(a) Compute $\dim(\ker T)$ and $\dim(\mathrm{Im} T)$.
(b) Describe a basis of $\mathbb{R}^n$ in which the matrix of T has the simplest possible block form. State that form.
(c) For which positive integers m does T^m have the *same* image as T? For which m does T^m have the *same* kernel as T?

Solution.
(a) $\ker T = \{0\}$ and $\mathrm{Im}\, T = \mathbb{R}^n$. Indeed, $(I + S)x = 0 \Rightarrow Sx = -x \overset{S}{\Rightarrow} 0 = -Sx \Rightarrow x = 0$.
(b) Choose a basis $\{x_i, u_i, y_j\}$ with $Sx_i = u_i$, $Su_i = 0$, $Sy_j = 0$; then $[T] = \mathrm{diag}(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{k \text{ copies}}, I_{n-2k})$.
(c) Since $S^2 = 0$, $(I + S)^m = I + mS$. As in (a), $\ker(I + mS) = \{0\}$ and the image is all of $\mathbb{R}^n$ for every $m \geqslant 1$.

Problem 4.3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Call T **decisive** if every nonzero vector in $\mathbb{R}^n$ belongs to exactly one of $\mathrm{Im}(T)$ or $\ker(T)$. Determine all decisive linear transformations.

Solution.
We begin with a lemma that forms the heart of the argument.

Lemma 4.4. *Let V and W be subspaces of $\mathbb{R}^n$ with $n \geqslant 2$. If $V \cup W = \mathbb{R}^n$, then $V = \mathbb{R}^n$ or $W = \mathbb{R}^n$.*

Proof. Suppose for contradiction that $V \neq \mathbb{R}^n$ and $W \neq \mathbb{R}^n$. Then there exist vectors $\vec{v} \in \mathbb{R}^n \backslash V$ and $\vec{w} \in \mathbb{R}^n \backslash W$. Since $V \cup W = \mathbb{R}^n$, we must have $\vec{v} \in W$ and $\vec{w} \in V$.
Consider the vector $\vec{u} = \vec{v} + \vec{w}$. Since $V \cup W = \mathbb{R}^n$, we have $\vec{u} \in V$ or $\vec{u} \in W$.

- If $\vec{u} \in V$, then $\vec{v} = \vec{u} - \vec{w} \in V$ (since $\vec{u} \in V$, $\vec{w} \in V$, and V is closed under subtraction). This contradicts $\vec{v} \notin V$.

- If $\vec{u} \in W$, then $\vec{w} = \vec{u} - \vec{v} \in W$ (since $\vec{u} \in W$, $\vec{v} \in W$, and W is closed under subtraction). This contradicts $\vec{w} \notin W$.

In either case we reach a contradiction, so $V = \mathbb{R}^n$ or $W = \mathbb{R}^n$. □

We now prove the main claim in two parts.

Part 1: $T = 0$ and T invertible are both decisive.

Case $T = 0$. Here $\ker(T) = \mathbb{R}^n$ and $\operatorname{Im}(T) = \{\vec{0}\}$. Every nonzero vector lies in $\ker(T)$ and does not lie in $\operatorname{Im}(T)$, so it belongs to exactly one of the two. Thus $T = 0$ is decisive.

Case T invertible. Here $\ker(T) = \{\vec{0}\}$ and $\operatorname{Im}(T) = \mathbb{R}^n$. Every nonzero vector lies in $\operatorname{Im}(T)$ and does not lie in $\ker(T)$, so it belongs to exactly one of the two. Thus any invertible T is decisive.

Part 2: No other T is decisive.

Suppose T is neither the zero map nor invertible. We must show T is not decisive.

Since T is not invertible, $\ker(T) \neq \{\vec{0}\}$, so there exists a nonzero vector in $\ker(T)$.

Since $T \neq 0$, $\operatorname{Im}(T) \neq \{\vec{0}\}$, so there exists a nonzero vector in $\operatorname{Im}(T)$.

Since T is not invertible, $\operatorname{Im}(T) \neq \mathbb{R}^n$ (by the rank-nullity theorem: $\ker(T) \neq \{\vec{0}\}$ implies $\operatorname{rank}(T) < n$, so $\operatorname{Im}(T) \subsetneq \mathbb{R}^n$).

Since $T \neq 0$, $\ker(T) \neq \mathbb{R}^n$ (otherwise T would be the zero map).

Now, for T to be decisive, every nonzero vector must lie in exactly one of $\operatorname{Im}(T)$ or $\ker(T)$. In particular, every nonzero vector must lie in *at least* one of them, which means

$$\mathbb{R}^n = \operatorname{Im}(T) \cup \ker(T).$$

(The zero vector belongs to both, and every nonzero vector belongs to at least one.)

But $\operatorname{Im}(T)$ and $\ker(T)$ are both proper subspaces of $\mathbb{R}^n$ (neither equals $\mathbb{R}^n$, as shown above). For $n = 1$, the only proper subspace of $\mathbb{R}^1$ is $\{\vec{0}\}$, so both $\operatorname{Im}(T)$ and $\ker(T)$ would be $\{\vec{0}\}$, contradicting $\operatorname{rank}(T) + \operatorname{nullity}(T) = 1$. For $n \geqslant 2$, Lemma 4.4 tells us that two proper subspaces cannot cover $\mathbb{R}^n$.

In all cases, $\operatorname{Im}(T) \cup \ker(T) \neq \mathbb{R}^n$, so some nonzero vector belongs to neither. Therefore T is not decisive. □